

# The Perils of Nominal Targets\*

Roc Armenter  
Federal Reserve Bank of Philadelphia<sup>†</sup>

February 17, 2016

## Abstract

A monetary authority can be committed to pursuing an inflation, price-level, or nominal-GDP target yet systematically fail to achieve the prescribed goal. Constrained by the zero lower bound on the policy rate, the monetary authority is unable to implement its objectives when private-sector expectations stray far enough from the target in the first place. Low-inflation expectations become self-fulfilling, resulting in an additional Markov equilibrium in which the monetary authority falls short of the nominal target, average output is below its efficient level and the policy rate is typically low. Introducing a stabilization goal for long-term nominal rates can implement a unique Markov equilibrium without fully compromising stabilization policy.

**Keywords:** Inflation targeting, Zero lower bound, Markov equilibria.

**JEL Codes:** E52, E58

---

\*This paper owes a great deal to conversations I had with Stefano Eusepi. I also thank Marco Bassetto, Jess Benhabib, Michael Dotsey, Pablo Guerron, Ben Lester, Andrew Postlewaite, Federico Ravenna, Lee Smith, and Ivan Werning, as well as participants at the NBER Monetary Economics meeting, Bank of Canada Annual Conference, Federal Reserve Bank of Philadelphia seminar, and the Federal Reserve Bank of Kansas City for comments and suggestions. The views expressed here do not necessarily reflect the views of the Federal Reserve Bank of Philadelphia or the Federal Reserve System. This paper is available free of charge at <http://www.philadelphiafed.org/research-and-data/publications/working-papers/>.

<sup>†</sup>10 Independence Mall; Philadelphia, PA 19106. Email: [roc.armenter@phil.frb.org](mailto:roc.armenter@phil.frb.org)

# 1 Introduction

Central banks across the developed world responded aggressively to the Great Recession by lowering their policy rates to historic lows. While a worse outcome may have been avoided, it is also fair to say that the ensuing recovery has been, at best, lackluster: Amid stagnant growth, employment has been slow to return to pre-recession levels and inflation has been persistently low. The subdued economic conditions have, in turn, led most central banks to keep policy rates at near zero—and several of those who dared to raise rates, like the European Central Bank or the Sveriges Riksbank, soon found themselves having to reverse course. Seven years in at the time of this writing, there are growing concerns that the combination of disappointing growth, suppressed inflation and very low rates will perpetuate.

Policymakers have often responded to these concerns by reiterating their commitment to their inflation targets, seeking to keep the private sector’s expectations anchored and to ease the pressure on real rates.<sup>1</sup> In practice, some countries have formalized their commitment to nominal targets by writing legislation detailing the monetary authority’s goals and putting in place procedures to ensure compliance, while other central banks rely instead on emphasizing their targets in their communications. The academic literature also views the central bank’s commitment to pursue the prescribed goals as indispensable for inflation targeting or, more broadly, for any monetary policy framework based on a nominal target.<sup>2</sup>

In this paper I show that a commitment by the monetary authority to pursue a nominal target—be for the inflation rate, the price level, or the nominal GDP—is a necessary, but not sufficient, condition to achieve such target. Attaining the prescribed target may simply be *unfeasible* when private-sector expectations depart from the target in the first place: The monetary authority, constrained by the zero lower bound (ZLB) on the policy rate, cannot steer the inflation rate, the price level or the nominal GDP back to target—no matter how *willing* the central bank is to do so. As result, multiple equilibria are pervasive, including an equilibrium in which the monetary authority systematically misses its target, policy rates are low, and average output is below its efficient level. The latter scenario stands in contrast with the existing literature, which is overwhelmingly supportive of nominal targets, and is reminiscent of the prevailing economic conditions since the Great Recession. My results also suggest that policymakers should combine their commitment to a nominal target with additional goals that remain attainable in all circumstances and are designed to counter low-inflation expectations, ideally without interfering with stabilization policy. I show how this could be accomplished with a stabilization goal for long-term nominal rates.

---

<sup>1</sup>For example, see Yellen (2015), Carney (2015) and Draghi (2014). On January 2013 the Bank of Japan announced a “price stability target” of 2 percent inflation.

<sup>2</sup>Svensson (2002) goes as far as to say that “an institutional commitment to inflation targeting appears essential for inflation targeting to have much meaning” (page 772).

The setting for my analysis is a simple, log-linearized New Keynesian model that observes the ZLB for the policy rate. In the spirit of Rogoff (1985) and Walsh (1995), society designs the monetary authority’s loss function to include the nominal target of choice, typically combined with a weight on output stabilization. As in practice, the central bank retains discretion in setting the policy rate.<sup>3</sup> There is no chance for society to revisit the central bank’s goals: Thus, the monetary authority’s willingness to pursue the specified goals is beyond any doubt. Policy is the outcome of the game between the monetary authority and the private sector and is thus time consistent in equilibrium.<sup>4</sup> While in theory society could specify a very rich set of goals for the central bank, the focus here will be squarely on targeting frameworks that have been implemented or are routinely put forward in policy debates.

The key to my analysis is how private-sector expectations constrain the feasibility set of the monetary authority—that is, its ability. Because of the ZLB, inflation expectations limit the range of real interest rates that the monetary authority can engineer, which in turn cap the inflation and output levels that can be implemented. If inflation expectations are low enough to start with, the monetary authority finds that it can only disprove the inflation and output expectations on the downside—and thus driving them both further away from their respective targets—by raising the policy rate above the ZLB. The least damaging option is to validate the private-sector expectations, and thus multiple equilibria arise.

The most complete set of results belong to inflation targeting. I show that a Markov equilibrium is, generically, not unique.<sup>5</sup> I also show that average inflation and output are strictly below the target and its efficient level, respectively, for at least one equilibrium; and weakly below for all equilibria. The distinct Markov equilibria are also ranked according to their average deviations from both the inflation target and the efficient output level. In the equilibrium with low inflation, the policy rate is often at the ZLB—though not necessarily all the time.

I must emphasize that the multiplicity of equilibria is pervasive. I make no additional assumptions on the model parameters and shocks are only required to follow a Markov process over a finite support. Perhaps most importantly, the equilibrium multiplicity does not depend on the value of the inflation target or whether output deviations are from its efficient level or a distorted steady state, and/or how the two are weighted. The intuition is clear cut. The underlying cause of the multiplicity is the monetary authority’s limited

---

<sup>3</sup>Svensson (1997) puts it succinctly: “The government delegates monetary policy to an instrument independent central bank that is assigned a particular loss function.” This is arguably how central banks actually operate—see Bernanke (2013), who describes it as “constrained discretion.” It is also the standard approach in the literature on the design of monetary policy frameworks.

<sup>4</sup>My analysis considers only Markov equilibria, that is, equilibria where allocations, policy. . . are stationary functions of the state of the economy. Thus I am abstracting from the possibility of history-dependent, time-dependent or other forms of non-stationary equilibria.

<sup>5</sup>It is possible in my model to have no or more than two equilibria.

ability to increase inflation—and not a lack of incentives to do so. Shifting the inflation target, re-weighting output and inflation deviations... only changes how the monetary authority evaluates outcomes, but not what it can achieve. The distinction between the monetary authority’s ability and willingness to pursue a goal instead guide us to consider additional goals rather than reinforcing the existing ones.

An additional, technical contribution of this paper is to provide an algorithm that computes all Markov equilibria. The algorithm is built upon the well-known problem of vertex enumeration in convex geometry and, while computationally demanding, can handle any general shock process with a finite support. Alternative approaches, like adaptive expectations, often fail to converge even if equilibria exist and are thus ill-equipped to establish equilibrium uniqueness or nonexistence. With the algorithm developed here I can provide an exhaustive robustness analysis regarding the number of equilibria as well as the determinants of equilibrium existence.

Price-level targeting does not resolve the equilibrium multiplicity despite introducing an endogenous state variable, namely, the price level (or its deviation from a predetermined path). There are always two equilibria in a nonstochastic economy. In one equilibrium, the monetary authority perfectly stabilizes the price level at the desired target (or targeted path). In the other equilibrium, the price level falls by an ever-increasing distance from the target and output stays below its efficient level. I also show that there are convergent global dynamics to the falling price-level equilibrium. The same set of results applies to nominal GDP targeting.

On a more positive note, inflation expectations are anchored on a single Markov equilibrium if the monetary authority has a strong goal for interest-rate stabilization. Once the penalty term on interest-rate deviations is large enough, the monetary authority will *increase* the policy rate—even though doing so will send both output and inflation further below target—when faced with deflationary expectations, disproving them and effectively ruling out a low-inflation Markov equilibrium.<sup>6</sup> Interest-rate stabilization, though, comes with its costs as stabilization policy strays further from optimal. Targeting a long-term nominal rate ameliorates the cost, since the long-term rate is mainly determined by inflation expectations—what the monetary authority needs to respond aggressively to anchor private-sector beliefs—and does not move much with short-term variation—leaving the monetary authority free to respond properly to shocks.<sup>7</sup> Arguably interest-rate volatility is already a concern of most central banks so stable long-run rates appears a reasonable goal for monetary policy.

I also explore whether inflation inertia, fiscal policy, or a temporary commitment to keep the policy rate low can implement a unique Markov equilibrium. Regarding inflation

---

<sup>6</sup>While sharing a similar aim, interest-rate stabilization is not related to Taylor rules with interest-rate smoothing, i.e., including a lagged policy-rate term. Indeed, the interest-rate stabilization is purely forward looking—and needs to be so if it is to uniquely pin down inflation expectations.

<sup>7</sup>Unfortunately, interest-rate stabilization only guarantees a unique Markov equilibrium; other, non-stationary equilibria may arise.

inertia, I find there are always two steady states in a non-stochastic economy. Even if past inflation was right on the target, there exists Markov equilibria where the economy converges to the ZLB as inflation slows down. Standard specifications for passive fiscal policy also accommodate both the stabilization and the low-inflation Markov equilibria; it is possible, though, to design fiscal policy rules that revert to being active if inflation expectations are low and act as an equilibrium selection device. Finally, I show that there are equilibrium dynamics leading to any Markov equilibrium following a temporary interest-rate peg.

There is no choice but to start the literature review with the paper whose title this paper pays homage to. Benhabib et al. (2001) show how inflation expectations are not anchored by an active Taylor rule once the ZLB is taken into account. The key difference is, of course, how the monetary authority decision is modeled. In Benhabib et al. (2001) the central bank follows a Taylor rule, while in my approach the monetary authority is a rational agent whose actions seek to maximize a set of goals. The set of policy designs spanned by varying the specification for the Taylor rule or the central bank's goals have points of contact but are, by large, non-overlapping. My approach obtains a “best response function” that specifies an action—that is, a policy rate—in response to the private-sector beliefs and shocks. This best response function is typically not recognizable as a simple Taylor rule, though in certain conditions it can provide a good approximation. In short, modeling policy as a delegation problem enables the study of popular policy frameworks, like inflation or price-level targeting, and opens up new possibilities, like interest-rate stabilization, that do not have a natural counter-part in a Taylor rule.

There is a vast, and still active, literature on inflation targeting, as well as a fast-growing one on price-level and nominal-GDP targeting, that has been very influential on the actual design of policy in many countries.<sup>8</sup> Most of the literature follows the approach here of modeling the monetary authority as a rational agent with a given set of goals—typically a nominal target combined with an output stabilization term. In particular, several papers in the literature have focused on the optimal constrained policy or design when a ZLB is present: See, for example, Jung et al. (2005), Adam and Billi (2007) and Nakov (2008), and more recently, Billi (2011) and Nakata and Schmidt (2014). These papers seek to characterize the optimal policy under discretion using a constrained maximization problem, which implicitly selects the equilibrium with highest social welfare when there are multiple equilibria—as my results indicate—but do not specify how the equilibrium selection occurs.<sup>9</sup> As far as I know, my paper is the first to point out that

---

<sup>8</sup>I cannot possibly hope to summarize the existing work here. Galí (2015) provides a quick introduction to the analysis of monetary policy frameworks in New Keynesian models. See also Clarida et al. (1999) and Bernanke and Woodford, eds (2005) for additional references, and Bordo and Siklos (2014) for an historical perspective.

<sup>9</sup>The situation is reminiscent of Ramsey equilibria in the analysis of optimal fiscal policy: There is no guarantee that the corresponding decentralization *uniquely* implements the desired outcome. See Bassetto (2005).

multiple equilibria are pervasive in monetary policy frameworks based on nominal targets once the ZLB is observed.

Most of the literature working with the ZLB and Taylor rules have ignored the “perils” noted by Benhabib et al. (2001), though this is rapidly changing. An early contribution is Eggertsson (2006) who also emphasizes the role that policy constraints play in the face of a large demand shock. More recently, researchers have started exploring whether the deflationary dynamics associated with an active Taylor can explain the aftermath of the Great Recession; see, for example, Schmidt-Grohe and Uribe (2013) and Piazza (2015). Aruoba and Schorfheide (2014) go as far as to estimate a rich New Keynesian model to see whether the data favor expectations-driven dynamics to explain U.S. and Japan data (they do for Japan). Mertens and Ravn (2014) characterize the impact of government expenditures upon output when the economy has been driven to the ZLB through a shift in expectations. These papers share a focus on the implications for fiscal policy.<sup>10</sup>

This paper is also closely related to the work on “expectations traps,” as branded by Chari et al. (1998). The standard setup is a game between the private sector and the monetary authority, which is assumed to maximize social welfare. Most of the literature has typically ignored the ZLB, yet multiple Markov equilibria still arise in several settings, as shown in Albanesi et al. (2003), Armenter and Bodenstein (2008), and Siu (2008), among others.<sup>11</sup> Armenter (2008) finds that very weak conditions are sufficient for multiple Markov equilibria to arise. The two key points of departure here from the literature are the focus on the design of the monetary authority’s objectives as well as the role of the ZLB—if the latter were to be dropped, the model presented here would be linear with a unique equilibrium.

The paper is organized as follows. Section 2 presents the model and the definition of equilibrium. Section 3 offers a walk-through of the key results in a simplified setting. Section 4 and 5 are devoted to inflation and price-level targeting, respectively. Section 6 explores one possibility to anchor inflation expectations based on interest-rate stabilization. The analytic results are complemented with some numerical exercises reported in Section 7. Finally Section 8 includes some extensions and Section 9 concludes.

---

<sup>10</sup>Cochrane (2015) is critical of such attempts, arguing that liquidity-trap predictions are quite sensitive to particular equilibrium choices.

<sup>11</sup>These papers posit the nominal interest rate as the sole monetary policy instrument, abstracting from money growth. This is not without loss of generality, as the set of equilibria may change depending on the instrument choice of the monetary authority, as the results in King and Wolman (2004) and Dotsey and Hornstein (2011) exemplify.

## 2 Model

### 2.1 The economy

Time is discrete and infinite,  $t = 0, 1, \dots$ . Let  $\pi_t, y_t$  and  $R_t$  denote the log-deviations in the inflation rate, output gap (or, simply, output), and nominal interest rate, respectively, from their efficient levels.<sup>12</sup>

The first equation of the model is the standard New Keynesian Phillips curve (NKPC)

$$\pi_t = \kappa y_t + \beta \pi_{t+1}^e + u_t, \quad (1)$$

where  $\pi_{t+1}^e$  denotes private-sector expectations for inflation at period  $t+1$ , formed at date  $t$ , and  $u_t$  is a cost-push shock. The slope of the NKPC and the intertemporal discount rate satisfy  $\kappa > 0$  and  $\beta \in (0, 1)$ , respectively.<sup>13</sup>

The second equation in the model is the intertemporal first-order condition, the Euler equation,

$$R_t = \sigma (y_{t+1}^e - y_t) + \pi_{t+1}^e + v_t, \quad (2)$$

where  $y_{t+1}^e$  denotes private-sector expectations for the output gap at period  $t+1$ , formed at date  $t$ ; and  $v_t$  is a real-rate shock. The intertemporal elasticity of substitution satisfies  $\sigma > 0$ .

The ZLB for the nominal interest rate is given by

$$R_t \geq -Z, \quad (3)$$

where  $Z > 0$  is the distance to the zero lower bound for the nominal interest rate, in logs, from the non-stochastic efficient level.<sup>14</sup>

Finally, society evaluates inflation and output gap deviations according to a quadratic loss function,

$$l_t = \pi_t^2 + \lambda y_t^2, \quad (4)$$

where the weight on output gap  $\lambda > 0$  is strictly positive.<sup>15</sup> As shown by Woodford (2003) and Benigno and Woodford (2004), the linear-quadratic approximation to the

---

<sup>12</sup>See Woodford (2003) and Galí (2015) for micro-foundations of the underlying non-linear economy. It is possible that additional equilibria arise in the non-linear economy, as argued by Braun et al. (2012). Log-linear deviations can also be computed from a distorted steady state.

<sup>13</sup>The NKPC specification assumes that nominal frictions in price setting are invariant to aggregate conditions. Siu (2008) finds that, depending on parameter values, there may be an additional equilibrium featuring high inflation in a model with endogenous price rigidity.

<sup>14</sup>Zero may not be the effective lower bound, as the success of several central banks at engineering negative nominal rates attest. My results do not depend on the value of  $Z$ : As long as there is a lower bound, be at zero or negative, equilibrium multiplicity will arise.

<sup>15</sup>It is possible to relate  $\lambda$  to the other parameters of the model using its micro-foundations and, indeed, I use such relationship when evaluating welfare in the numerical exercises.

social welfare function around the non-stochastic efficient variables is justified if there are no distortions under price stability—in the parlance of the literature, there are no permanent differences between the efficient and natural levels of output. I assume this to be the case, which has the additional advantage to ensure that the efficient allocation is always a non-stochastic steady state for all the monetary frameworks considered here.<sup>16</sup> Total welfare is given by the loss function  $L_t = E_t \sum_{j=t}^{\infty} \beta^{j-t} l_j$ .

The description of the economy is closed by the shock specification. Let  $s_t = \{u_t, v_t\}$ . The exogenous state is governed by a first-order Markov process  $F(s'|s)$  over finite support  $S$ , with cardinality  $n$ . I assume that  $F(s'|s) > 0$  for all pairs  $\{s, s'\} \in S^2$  and that there is a unique ergodic distribution such that the unconditional mean of both shocks is zero. Let  $F^j(s'|s)$  be the conditional distribution  $j$  steps ahead. The realization of  $s_t$  is observed by all agents—including the monetary authority—at the beginning of period  $t$ .

## 2.2 The monetary authority

I model the monetary framework as one of delegation. Society determines the central bank’s objectives by designing its loss function,  $l^p(\pi, y)$ , which needs not coincide with society’s welfare function,  $l_t$ . The monetary authority is instrument independent, that is, it retains discretion to set the nominal interest rate,  $R_t$ , or policy rate. This approach traces back to Rogoff (1985) and has been prevalent in the literature on nominal targets.<sup>17</sup> I assume the monetary authority has the same discounting as society, and thus total welfare loss is simply given by  $L_t^p = \sum_{j=t}^{\infty} \beta^{j-t} l_j^p$ .

A key assumption is that society is perfectly committed to the central bank’s objectives, that is, the loss function  $l^p(\pi, y)$  is not revised at any date. In this sense, the monetary authority’s pursuit of the objectives prescribed by society is beyond any doubt, that is, there is no question of the central bank’s *willingness* at all dates and states of the world to set the policy rate in the manner most congruent with the prescribed targets.

## 2.3 Markov equilibrium

All is set for the equilibrium definition. Formally, the model can be viewed as a game between a strategic player, the monetary authority, and a nonstrategic player, the private sector. I choose to restrict the set of equilibria under analysis to Markov equilibria, that is, equilibria so that allocations and policy are a stationary function of the state of the

---

<sup>16</sup>For example, there will be no inflationary bias when we consider policy discretion. Benigno and Woodford (2005) show how to derive a linear-quadratic approximation for the case of a distorted steady state.

<sup>17</sup>See Walsh (1995) for a careful formalization of the delegation problem. It is also possible to state instead a policy rule designed to implement the desired target: see Svensson (2002) for a discussion. These rules, though, rarely take a simple form.



economy only.<sup>18</sup> The Markov refinement rules all forms of non-stationary equilibria, e.g., sunspot equilibria, history- or time-dependent equilibria. . . I am thus focusing on a very narrow definition of equilibrium yet, as we shall see, equilibrium multiplicity is pervasive.<sup>19</sup>

**Definition 1.** *A Markov equilibrium is a set of vectors in  $\mathfrak{R}^n$ ,*

$$\{R, \pi, y, \pi^e, y^e\},$$

*such that for all  $s \in S$ :*

1. *Equilibrium equations (1)-(3) are satisfied,*
2. *Rational expectations hold,  $\pi(s) = \pi^e(s), y(s) = y^e(s)$ ,*
3. *The nominal interest rate solves*

$$\min_R L_t^p$$

*subject to (1)-(3) and taken as given private-sector expectations,  $\pi^e, y^e$ .*

Since rational expectations imply  $\pi(s) = \pi^e(s), y(s) = y^e(s)$ , I annotate the description of a Markov equilibrium to three vectors,  $\{R, \pi, y\}$ , from now on.

## 2.4 Generic economies

Let  $\xi$  be a vector collecting all the structural parameters in the model

$$\xi = \{\kappa, \beta, \sigma, Z, F, S\},$$

and  $\Xi$  the corresponding set of admissible parameter vectors constructed from the parameter restrictions given in the description of the model.<sup>20</sup> A property of an economy  $\xi \in \Xi$  is “generic” if it holds in a neighborhood of  $\xi$ ; that is, there exists  $\delta > 0$  such that for any  $\xi' \in \Xi$  with  $\|\xi - \xi'\| < \delta$  the property is also satisfied. Conversely, a property of economy  $\xi$  is not generic if there is an arbitrarily close economy  $\xi' \in \Xi$  that does not satisfy it.

For all purposes here, whether a property is generic or not is settled on the basis of the number of solutions a linear equation system has. A system of  $n$  distinct linear equations will generically have exactly one solution. If  $n$  equations are combined with an additional condition, then generically the system will have no solution. Intuitively, a system of  $n + 1$  equations with  $n$  unknowns that has a solution implies that the system has rank  $n$  and thus implies an exact relationship between coefficients—that is, parameters in the model. It is thus a measure-zero set and thus not generic.

<sup>18</sup>When studying price-level targeting it will be necessary to restate the equilibrium definition, as the state of the economy will then include the price level.

<sup>19</sup>Markov equilibria are usually the equilibrium definition of choice in the literature on expectation traps; see Albanesi et al. (2003) and Armenter (2008), for example.

<sup>20</sup>From these restrictions it is also immediate that  $\Xi$  is a convex set.

### 3 Inflation expectations and the monetary authority's options

The key to this paper's results is how private-sector expectations, together with the ZLB, constrain the range of inflation rates and output gaps the monetary authority can achieve. Under some simplifying assumptions, the correspondence between the private-sector expectations and the monetary authority's feasibility set is easily derived. I show how low-inflation expectations render the inflation or price-level target unfeasible so the central bank, despite being committed to pursuing its targets, would systematically fail to achieve them. I also include a brief discussion on how interest-rate stabilization can anchor expectations.

Abstract from shocks and assume the private sector believes inflation to be constant at  $\pi^e$  from period  $t + 1$  onward. The private-sector expectation for the output gap must be consistent with the inflation expectations, so it satisfies (1):

$$y^e = \frac{1 - \beta}{\kappa} \pi^e$$

and is also constant over time from period  $t + 1$  onward. Let us now characterize the set of present-period inflation rates and output levels  $\pi_t, y_t$  that the monetary authority can engineer given private-sector expectations. The ZLB (3) constrains the policy rate. Substituting the Euler equation (2), we obtain

$$R_t = \sigma (y^e - y_t) + \pi^e \geq -Z.$$

Inflation expectations are given, so the real rate will move with the policy rate one-to-one, which in turn will induce the output gap to adjust in the present period. However, the real rate inherits the ZLB, with its exact lower bound pinned down by inflation expectations. If the inflation expectations are high, the real rate can go very low. But if they are low, then the real rate may not decrease much:

$$r_t = \sigma (y^e - y_t) \geq -Z - \pi^e.$$

Since the expectation for the next-period output gap is also given, we also have an upper bound on the output gap that can be achieved:

$$\begin{aligned} y_t &\leq \frac{1}{\sigma} (Z + \pi^e) + x^e \\ &\leq \frac{1}{\sigma} Z + \left( \frac{1}{\sigma} + \frac{1 - \beta}{\kappa} \right) \pi^e, \end{aligned}$$

where in the second line I substituted for the output expectation solved before. The upper bound in the output gap at period  $t$ , in turn, implies an upper bound on inflation at date  $t$ , as given by (1):

$$\pi_t \leq \frac{\kappa}{\sigma} Z + \left( \frac{\kappa}{\sigma} + 1 \right) \pi^e.$$

Thus, the ZLB imposes an upper bound both on the inflation and on the output gap that the monetary authority can engineer. The bound is tighter the lower inflation expectations are—as the real interest rate cannot drop below the ZLB minus inflation expectations.

To summarize, lower inflation expectations can drastically cap the range of inflation and output gap values the monetary authority can implement. To see how this naturally leads to multiple equilibria under policy discretion, consider a very simple specification for the monetary authority’s loss function,  $l_t^p = \pi_t^2$ , that is, the sole objective of monetary policy is to stabilize inflation.

When the private-sector expects inflation and output to be fully stabilized from the next period onward,  $\pi^e = 0$ , the monetary authority can stabilize inflation and output in the present period as well,  $\pi_t = y_t = 0$ , simply by setting  $R_t = 0$ . Clearly that is the choice of a monetary authority under a strict inflation-targeting mandate, for  $\pi_t = 0$  is actually the global maximum. It is easy to see how this readily generalizes: As long as the monetary authority’s objectives value full stabilization—and it would be extravagant not to—this will be the basis for the “good” Markov equilibrium.

Let us turn to the possibility that inflation expectations are quite low to start with—indeed, low enough so that the monetary authority finds that it cannot implement an inflation rate that is higher than what the private sector expected,  $\pi_t \leq \pi^e$ , that is,

$$\pi_t \leq \frac{\kappa}{\sigma} Z + \left( \frac{\kappa}{\sigma} + 1 \right) \pi^e = \pi^e,$$

which readily returns  $\pi^e = -Z < 0$ . In other words, the central bank’s only chance to disavow inflation expectations is on the downside, that is, further away from full stabilization. No matter how committed the monetary authority it is to defend its inflation target, the best it can do is to validate the inflation expectations by setting the nominal interest rate as low as possible, that is, right at the ZLB. This would be the basis of the low-inflation, or “bad,” Markov equilibrium. It is also irrelevant how output and inflation deviations are weighted: the monetary authority’s options regarding the output gap are also bounded above by the private-sector expectations,  $y_t \leq y^e$ . Regardless of the monetary authority’s policy rate decision, inflation and output will *both* be below private-sector expectation and their efficient levels. The monetary authority will still do its best to minimize the deviations, which is to set the nominal rate at zero and validate the private-sector expectations.

In a stochastic economy the feasibility set of the monetary authority is not so tightly characterized, as shocks can give the central bank some space to maneuver the policy rate. In Section 4, I show that equilibrium multiplicity arises under any shock process.<sup>21</sup> The dynamics of the policy rate are also somewhat more complex. The policy rate may be at the ZLB in the stabilization equilibrium, though typically only briefly. In the low-inflation equilibrium the economy spends long spells at the liquidity trap, though the policy rate

---

<sup>21</sup>Unfortunately, it is also possible to have no equilibria or to have more than two.

may also rise above the ZLB for substantial periods. Importantly, I am able to prove that both output and inflation are, on average, below their targets independently of the underlying shock process.

Turning to price-level targeting, it is clear that if the price level is at or below target, the situation is essentially identical to inflation targeting: Faced with low-inflation expectations, the monetary authority will fall short of both the output and price-level targets. However, if the price level is *above* target, the monetary authority may actually want to lower inflation below the private-sector expectations. Does this rule out the low-inflation equilibrium? Unfortunately, it does not. I show in Section 5 that there is an equilibrium such that, for any price level, the economy eventually converges to the ZLB. Along the path, output also stays below target.

What does it take to rule out the low-inflation equilibrium? To disprove low-inflation expectations, the monetary authority should be willing to decrease inflation and output further. Such a response can be induced if we include a stabilization goal for the policy rate—i.e., a penalty term for policy rate deviations. If the weight on the rate deviations is large enough in policy loss function, the monetary authority will increase the policy rate despite depressing output and inflation further in doing so. The same logic applies if any nominal rate, medium or long term, is used. In Section 6, I provide a threshold condition on the weight on interest-rate stabilization that it is sufficient to anchor expectations; that is, to implement a unique Markov equilibrium. I also argue that the stabilization goal is best set on the long-term nominal rate, which allows the monetary authority to pin down inflation expectations without fully compromising stabilization policy.

## 4 Inflation targeting

Inflation targeting was introduced in New Zealand in 1990. Close to 25 years later, it is the monetary framework of choice among developed economies. Researchers have found strong theoretical foundations to back up inflation targeting’s success. Bernanke and Mishkin (1997) described inflation targeting as “constrained discretion,” where the monetary authority operates with well-defined goals under a high degree of transparency and accountability. In this context, inflation targeting has been repeatedly shown to be preferable to “full” discretion.<sup>22</sup>

The literature is unanimous that an institutional commitment is a necessary condition for inflation targeting. Svensson (2002) goes as far as to consider commitment as “essential for inflation targeting to have much meaning.”<sup>23</sup> The results below show that, unfortunately, such a commitment is not sufficient to ensure that an inflation-targeting central bank achieves its goals.

---

<sup>22</sup>Svensson (1997) is credited as a keystone in the literature, yet research on inflation targeting remains active on several fronts: see Bernanke and Woodford, eds (2005) for an overview.

<sup>23</sup>Page 772.

## 4.1 The monetary authority’s objectives

I adopt the standard specification for a flexible inflation-targeting central bank,

$$l_t^p = \pi_t^2 + \psi y_t^2, \tag{5}$$

where  $\psi \geq 0$ . The case  $\psi = \lambda$  corresponds to a central bank pursuing exactly society’s welfare, while the case  $\psi = 0$  is what has been termed an “inflation nutter.” Because of the underlying assumptions in the economy, there is no “inflationary bias” problem as in Barro and Gordon (1983). There is, however, a “stabilization bias,” as defined in Svensson (1997) and Clarida et al. (1999), which provides a rationale for why society would like to set  $\psi < \lambda$ .<sup>24</sup>

The necessary and sufficient first-order condition associated with the monetary authority’s problem is simply

$$\kappa \pi(s) + \psi y(s) \leq 0 \tag{6}$$

with strict equality if  $R(s) > -Z$ . Equation (6) summarizes what the monetary authority intends to achieve and what it will actually achieve. Ignore first the possibility that the ZLB is binding and thus the first-order condition will hold with strict equality. Real-interest rate shocks pose no obstacle to fully stabilizing output and inflation. Cost-push shocks, though, will force the monetary authority to trade off inflation deviations for output deviations: It will do so at rate  $\psi/\kappa$ , where  $\kappa$  is the marginal rate of “transformation” of inflation to output deviations and  $\psi$  is the corresponding marginal rate of substitution as dictated by the policy loss function. The monetary authority will end up missing both targets, but never on the same side. That is, whenever inflation is above target, output will be below, and vice versa.

The monetary authority may also miss its targets because of the ZLB. Now both cost-push and real-interest rate shocks can create inflation and output deviations. Moreover, it is possible that both inflation and output are simultaneously below target; that is, (6) is slacked. These possibilities and their resulting dynamics have been extensively studied elsewhere, most recently in medium- to large-scale macroeconomic models.<sup>25</sup> The ZLB, though, also opens the possibility that the private-sector expectations, by themselves, render the monetary authority unable to achieve its prescribed goals.

The first-order condition (6) is sometimes introduced in the set of equilibrium equations as a “targeting rule.” If ZLB is ignored, a targeting rule is equivalent to the delegation problem proposed above. However, if we wish to respect the ZLB, great care is needed to make sure the targeting rule is slacked, that is, it holds with inequality. Imposing the targeting with equality may implicitly select an equilibrium where the ZLB does not bind, ignoring the existence of other equilibria where it does bind.

---

<sup>24</sup>The ZLB may offer a rationale for a *less* conservative policymaker: See Nakata and Schmidt (2014).

<sup>25</sup>See Fernandez-Villaverde et al. (2012) and Lopez-Salido et al. (2010), among many others.

I should also note that the policy described by (6) cannot be implemented with a simple Taylor rule—even if the ZLB is ignored. Taking (6) with strict equality, and substituting into the Euler equation (2), it is clear that the policy rate should be a function of either the expected output gap or the two-step ahead inflation expectation: Neither term is commonly featured in Taylor rule specifications. That said, the Taylor rule can provide a reasonable approximation under certain conditions.<sup>26</sup>

## 4.2 Equilibrium characterization

The main challenge is that we do not know ex-ante which states have the nominal rate binding at the ZLB. In principle, one can proceed by guessing the states such that the ZLB is binding, solve for the nominal rate at every state using the corresponding equilibrium equation as given by the guess, and then verify whether the ZLB (3) is indeed binding at the states conjectured to do so—and none more. This approach is obviously ill-suited for analytic results.

Instead I express the model exclusively in terms of actual and expected inflation dynamics, recasting the ZLB condition as a simple nonlinear equation that can be solved simultaneously with the remaining equations. From the first-order condition of the monetary authority (6) and the Euler equation (2), combined with the ZLB, I derive two equilibrium *inequalities* for inflation. To close the equilibrium I only need to make sure that at each state one of the inequalities is holding with strict equality. The latter equilibrium condition is easily implemented by taking the intersection of the two inequalities, which also automatically determines whether the ZLB is binding or not.

First I solve for the output gap. From (1), the output gap can be expressed as

$$\kappa y(s) = \pi(s) - \beta E_1(\pi|s) - u,$$

where

$$E_1(\pi|s) = \sum_S \pi(s') F(s'|s)$$

is the one-step-ahead inflation expectation.

In the ZLB condition (3) I substitute using the Euler equation (2) as well as the above expression for the output gap to obtain

$$\sigma(E_1(y|s) - y(s)) + E_1(\pi|s) + v \geq -Z,$$

where  $E_1(y|s)$  follows the same shorthand notation as  $E_1(\pi|s)$ . Let  $\nu = \kappa\sigma^{-1}$ . Substituting for actual and expected output, we arrive at

$$\pi(s) \leq \pi^b(s) \equiv (1 + \nu + \beta) E_1(\pi|s) - \beta E_2(\pi|s) + A(s), \quad (7)$$

---

<sup>26</sup>See Woodford (2001).

where  $A(s) = u - E_1(u|s) + \nu v + \nu Z$ , and

$$E_2(\pi|s) = \sum_s E_1(\pi|s') F(s'|s)$$

is the two-steps-ahead inflation expectation. I have thus translated the ZLB on the nominal rate into an inequality stating an upper bound for the inflation rate, (7). If the ZLB is indeed binding, then (7) will hold with strict equality,  $\pi(s) = \pi^b(s)$ .

The first-order condition of the monetary authority, (6), provides the other key equilibrium condition. Note that the first-order condition is indeed an inequality, to hold with strict equality whenever the ZLB is not binding—that is, precisely when (7) is slack. Substituting again for the output gap, I obtain

$$\pi(s) \leq \pi^u(s) \equiv \frac{\psi}{\kappa^2 + \psi} (\beta E_1(\pi|s) + u). \quad (8)$$

At every state, either (7) or (8) must hold with strict equality. Since both conditions are effectively an upper bound on the present inflation rate, whether the ZLB is binding or not is resolved by taking the minimum of both inequalities,

$$\pi(s) = \min \{ \pi^u(s), \pi^b(s) \}. \quad (9)$$

The equilibrium inflation vector  $\pi$  is thus characterized by a system of  $n$  piece-wise linear equations from a total of  $2n$  linear functions. The condition that is in effect at each state given the equilibrium solution determines whether the ZLB is binding or not. There is no difficulty in retrieving the equilibrium values for the interest rate and the output gap.

### 4.3 Equilibrium multiplicity

For the centerpiece result of this paper, I show that there is, generically, no unique equilibrium under inflation targeting. The result is surprisingly broad, as it does not require any further restriction on parameters and encompasses any shock process we wish to impose. It also applies to all inflation-targeting regimes, from a central bank with very high output weight to an inflation nutter, that is, any policy loss function  $\psi \geq 0$ ; or for alternative values of the ZLB  $Z > 0$ .

An analytic result is particularly useful because the system of nonlinear equations described by (9) poses some challenges. There may be no equilibrium at all for some economies. If they exist, equilibria can be characterized as fixed points of a vector-valued function, yet the accompanying mapping is not well behaved. Numerical methods inherit these difficulties: Iteration by adaptive expectations, for example, may fail to converge even if an equilibrium does exist.

**Proposition 4.1.** *Let  $\{\pi^*, R^*, y^*\}$  be a Markov equilibrium. Then, generically, there exists at least an additional Markov equilibrium,  $\{\tilde{\pi}, \tilde{R}, \tilde{y}\}$ , with  $\tilde{\pi} \neq \pi^*$ .*

*Proof.* In the appendix □

The proof takes a novel approach that exploits the piece-wise linear structure of the equilibrium equations. For each state, the two equilibrium conditions (7) and (8) can be viewed as describing two half-spaces in  $\mathfrak{R}^n$ , namely,

$$H^u(s) = \{x \in \mathfrak{R}^n : x \geq (1 + \nu + \beta) E_1(x|s) - \beta E_2(x|s) + A(s)\}$$

and

$$H^b(s) = \left\{ x \in \mathfrak{R}^n : x \geq \frac{\psi}{\kappa^2 + \psi} (\beta E_1(x|s) + u) \right\}.$$

Note that their intersection naturally satisfies (9). I show that the  $2n$  half-spaces describe a solid polytope in  $\mathfrak{R}^n$ : A Markov equilibrium must be a vertex, as it is a point where the boundaries of  $n$  of those half-spaces intersect.<sup>27</sup> Not all vertexes are equilibria, though. There are “kinks” defined by the intersection of the boundary of two half-spaces corresponding to the same equilibrium equation. The final step of the proof shows that it is not possible that all vertexes but one are “kinks,” and thus there are at least two Markov equilibria.

Proposition 4.1 does not require any additional assumptions on parameters or the stochastic process for shocks, and can also be readily extended to economies with non-zero mean shocks or steady-state distortions. Perhaps more importantly, Proposition 4.1 also applies to a more general specification for the monetary authority’s targets, that is, loss functions of the form:

$$l_t^p = (\pi_t - \bar{\pi})^2 + \psi (y_t - \bar{y})^2$$

where the monetary authority’s targets for inflation and output,  $\bar{\pi}, \bar{y}$ , can be either positive or negative, that is, above or below their efficient counterparts. Thus shifting the inflation or output targets for the monetary authority does not resolve the inherent multiplicity problem. As discussed earlier, the underlying cause is the monetary authority’s ability, which does not change with the targets. Technically, the extensions discussed above represent a simple geometric translation of the half-spaces  $H^u$  and  $H^b$ , as they simply introduce constant terms to the equilibrium equations (7) and (8).<sup>28</sup>

One shortcoming of Proposition 4.1 is that while it rules out a unique equilibrium, it does not establish the actual number of equilibria—no equilibria being a possibility. Fortunately, the proof of Proposition 4.1 shows that characterizing the full set of equilibria is a vertex enumeration problem, for which readily available algorithms already exist. While computationally expensive, these algorithms guarantee that no Markov equilibria will be missed and thus that can be used to establish equilibrium nonexistence or uniqueness as well.

---

<sup>27</sup>Incidentally, this proves that each Markov equilibrium is locally unique.

<sup>28</sup>Similarly, the result does not depend on the value of  $Z$ , so the model can encompass negative effective lower bounds on nominal rates.



While it may not be immediately apparent, Proposition 4.1 does not apply to economies with i.i.d. shocks, as such restriction is “not generic” from the point of view of the general set of possible shock processes. This is no major problem as an equivalent result exists for i.i.d. economies, with a much simpler proof and the added benefit that the number of Markov equilibria is either zero or two.

**Proposition 4.2.** *Let  $F(s'|s) = F(s')$  for all  $s, s' \in S$ . Then, generically, there is either no Markov equilibrium or two Markov equilibria.*

*Proof.* In the appendix □

## 4.4 Equilibrium properties

The next set of results provides some key properties regarding average inflation, output and the interest rate across Markov equilibria. As we shall see, there exists an equilibrium in which output and inflation are strictly below target, and interest rates are abnormally low—arguably an accurate description of the scenario many developed countries are currently concerned about.

Let  $Ex$  denote the unconditional expectation (or average) for  $x = \{\pi, y, R\}$ , given  $Ex \equiv \sum_{s \in S} f(s)x(s)$ , where  $f \in \Delta^n$  is the ergodic distribution over  $S$  generated by the Markov process  $F$ .

The first result shows that inflation, output and the interest rate are, on average, equal to their target or below in a Markov equilibrium. Perhaps more importantly, there is, generically, a Markov equilibrium such that average output is *strictly* below its efficient level and, despite the monetary authority setting interest rates at low levels, average inflation fall shorts of the target.

**Proposition 4.3.** *For any Markov equilibrium  $\{\pi^*, y^*, R^*\}$ , the unconditional expectations are non-positive,  $E\pi^* \leq 0$ ,  $Ey^* \leq 0$  and  $ER^* \leq 0$ . Moreover, for at least one Markov equilibrium, the unconditional expectations are, generically, strictly negative  $E\pi^* < 0$ ,  $Ey^* < 0$  and  $ER^* < 0$ .*

The intuition behind Proposition 4.3 is simple. For a Markov equilibrium such that the ZLB does not bind at any state, average inflation, output and interest rate will be exactly on target, i.e.,  $E\pi = Ey = ER = 0$ .<sup>29</sup> In such equilibrium, real-rate shocks can be and are completely stabilized. Consider now a Markov equilibrium such that the ZLB binds occasionally. In such occasions, the monetary authority cannot counter downward real-rate shocks which push both inflation and output below target. Hence the downward bias on both inflation and output as well as the interest rate. The existence of at least one Markov equilibrium with strictly-negative average follows from Proposition

---

<sup>29</sup>Indeed, the converse is also true: If either average inflation, output, or interest rate are equal to the target, then the ZLB is, generically, not binding.

4.1 as, generically, no distinct Markov equilibria  $\{\pi^*, y^*, R^*\}, \{\pi^{**}, y^{**}, R^{**}\}$  can satisfy  $E\pi^* = E\pi^{**}$ .

Proposition 4.3 does not tell us much on how different Markov equilibria compare with each other if both have average inflation below target. Fortunately, the following Corollary strengthens the result substantially, showing that Markov equilibria with lower average inflation also feature lower average output and interest rates.<sup>30</sup>

**Corollary 4.1.** *Let  $\{\pi^*, y^*, R^*\}, \{\pi^{**}, y^{**}, R^{**}\}$  be two distinct Markov equilibria with  $E\pi^* < E\pi^{**}$ . Then  $Ey^* < Ey^{**}$  and  $ER^* < ER^{**}$ .*

It is worth remarking on the properties of the output gap established by Proposition 4.3 and Corollary 4.1. Had we found that average output was at or above its efficient level, we would be discussing a failure in stabilization policy, whose welfare implications would hinge on how society trades off inflation and output deviations—and a failure likely to be ameliorated by tweaking the monetary authority’s objectives, either by means of moving the target levels as in Barro and Gordon (1983) or the weight on inflation stabilization as in Rogoff (1985). Instead, *both* average output and inflation are below target—a situation undesirable no matter how society weighs inflation and output deviations as well as one that cannot be solved by tweaking the existing objectives. Similarly, Corollary 4.1 makes it clear that Markov equilibria are not just distributed along some inflation-output frontier, with equilibria with large deviations in average inflation possibly having smaller ones for average output. Instead, Markov equilibria can be, generically, strictly ranked according to their average deviations from target independently on whether we look only at inflation, output or any weighted average of them.

The results provide a reasonably complete picture of average inflation, output and interest rates for the set of Markov equilibria. It is unfortunately difficult to characterize inflation or output dynamics any further for a general shock process. The case of i.i.d. shocks, albeit more restrictive, delivers a stark comparison about inflation dynamics.

**Proposition 4.4.** *Let  $F(s'|s) = F(s')$  for all  $s, s' \in S$ . Let  $\{\pi^*, y^*, R^*\}, \{\pi^{**}, y^{**}, R^{**}\}$  be two distinct Markov equilibria and, without loss of generality, let  $E\pi^* < E\pi^{**}$ . Then  $\pi^*(s) < \pi^{**}(s)$  for all  $s \in S$ .*

In other words, the lower average inflation is brought in by lower inflation at all states—not just those with large shocks or those when the ZLB is binding. A similar result for output does not hold, as the NKPC (1) dictates that output dynamics respond to realized and expected inflation in opposite directions. The interplay between output and future inflation is also behind the lack of analytic results for more general shock processes. Numerical results, though, suggest that the pattern for inflation dynamics stand for all but unusual shock processes.

---

<sup>30</sup>Recall that Proposition 4.1 establishes that there are, generically, either no or multiple distinct Markov equilibria. The case in which two distinct equilibria have the same average,  $E\pi^* = E\pi^{**} < 0$ , is not generic.

## 5 Price-level targeting

Inflation targeting may not anchor private-sector expectations. Perhaps price-level targeting fares better, as it pins down policy to a state variable? Unfortunately, it does not.

Svensson (1999) was among the first to analyze price-level targeting at par with inflation targeting. In a key contribution, Vestin (2006) shows that a monetary authority committed to pursuing a price-level target would actually mimic the optimal policy. His analysis ignored the ZLB on the policy rate: It turns out that what really sets apart price-level targeting is its ability to approximate the optimal policy at the ZLB, as first pointed out by Eggertsson and Woodford (2003).<sup>31</sup>

### 5.1 The monetary authority

The price level, in terms of deviations from some steady-state underlying trend, is given by

$$p_t = \pi_t + p_{t-1}. \quad (10)$$

Note the timing regarding the definition of the price level:  $p_t$  is the end-of-period  $t$  price level. The price-level target is implemented by endowing the monetary authority with an objective function

$$l_t^p = p_t^2 + \psi y_t^2, \quad (11)$$

where  $\psi \geq 0$  allows for flexible price-level targeting, i.e., the monetary authority also has a stabilization goal for output. The initial value for the price level,  $p_0$ , is taken as given. The specification (11) for the price-level targeting nests nominal GDP targeting as a special case with  $\psi = 1$ .<sup>32</sup>

The price level is now an endogenous state variable in this economy.<sup>33</sup> As a result, the monetary authority's decision has an explicit intertemporal dimension: When setting the nominal interest rate in the present period, the monetary authority realizes that it will impact the price level, which will, in turn, shape future rate decisions. The analysis is further complicated because one should not expect the continuation value function to be differentiable with respect to the price level. In order to be able to characterize the law of motion for the price level, I have no choice but to restrict the analysis to perfect-foresight economies.

---

<sup>31</sup>See also Wolman (2005), Jung et al. (2005), Adam and Billi (2006), and Nakov (2008). There is very limited empirical evidence on price-level targeting because it has rarely been implemented. An exception is Berg and Jonung (1999), who provide an account of Sweden's experience with price-level targeting in the 1930s.

<sup>32</sup>Some readers may prefer a more literal interpretation of nominal GDP targeting, with the monetary authority's loss function defined as  $l_t^p = (p_t + y_t)^2$ . The same results below apply to both specifications: There are only minor changes in the proof, detailed in the Appendix.

<sup>33</sup>The appendix contains an updated definition of a Markov equilibrium.

## 5.2 Results

Here I show that there is a Markov equilibrium such that, for any initial price level,  $p_0$ , the nominal interest rate converges to the ZLB in finite time, with output and the price level falling permanently below target after that. There is, of course, a more benign Markov equilibrium in which inflation, output, and the price level converge to full stabilization—see Vestin (2006) and Eggertsson and Woodford (2003) for details.

**Proposition 5.1.** *Consider a perfect-foresight economy,  $F(\{0, 0\}) = 1$ . There exists a Markov equilibrium and a finite time  $t^*(p_0)$  such that, for all  $t \geq t^*(p_0)$ , the nominal interest rate is at the ZLB,  $R_t = -Z$ , and both inflation and output are below target,  $\pi_t, y_t < 0$ .*

*Proof.* In the appendix □

As discussed in Section 3, the steady-state inflation and output levels are the same as in the low-inflation equilibrium under inflation targeting. The key contribution of Proposition 5.1 is to show that there is global convergence to the low-inflation steady state. The proof proceeds by construction, conjecturing an upper bound on the law of motion for the price level as well as the aforementioned steady-state level for inflation and output. Note that since inflation is systematically below target, the price level is drifting further and further away from its target. Since the policy loss function is unbounded, the proof must also show that the monetary authority’s value function remains well defined.

The result in Proposition 5.1 stands in contrast to some well-known results in the literature. Eggertsson and Woodford (2003) write that “eventually it [the central bank] will be able to hit the target” when discussing price-level targeting.<sup>34</sup> What is behind such contradicting conclusions? In Eggertsson and Woodford (2003), the economy finds itself against the ZLB due to a real-interest rate shock. Once the shock unwinds, the economy lifts from the ZLB and the monetary authority has no trouble returning the price level to the desired target. In my analysis, though, the culprit of the economy’s dire situation is private-sector expectations that do not unwind, and thus the monetary authority never has a chance to return the price level (or inflation) to target.<sup>35</sup>

## 6 Interest-rate stabilization

In this section I show that private-sector expectations can be pinned down by a unique Markov equilibrium if the monetary authority has a stabilization goal for nominal interest rates. Such goal appears reasonable—arguably, central banks do indeed care about stable interest rates—and relatively simple to communicate and evaluate.

---

<sup>34</sup>Page 184.

<sup>35</sup>See Section 8 for an extended discussion of Eggertsson and Woodford (2003).

## 6.1 The monetary authority's objectives

Let  $R_t^j$  be the nominal rate return paid at maturity after  $j \geq 1$  periods, in annualized rate. The policy rate,  $R_t$ , is simply  $j = 1$ . For longer duration bonds, the return must satisfy the arbitrage condition

$$R_t^j = \frac{1}{j} E_t \left( \sum_{i=0}^{j-1} R_{t+i} \right). \quad (12)$$

Say the monetary authority's inflation and output targets are now combined with a penalty term for deviations in the nominal rate, for some horizon  $j$ , from its steady-state level—in other words, the stabilization of the nominal interest rate is made an explicit goal of the central bank. The monetary authority's loss function is specified as

$$l_t^p = \pi_t^2 + \psi y_t^2 + \rho \left( R_t^j \sqrt{j} \right)^2, \quad (13)$$

where  $\psi, \rho \geq 0$  and the term  $\sqrt{j}$  is a convenient normalization. I should emphasize that the nominal rate stabilization goal is taken to be a deviation from the social welfare, which remains given by (4). As such, it is legitimate to ask whether society would indeed be able to commit to such goal for the monetary authority.<sup>36</sup>

Before stating the necessary and sufficient first-order condition associated with the monetary authority's problem, I solve for the nominal rate using (12) and (2) for the whole horizon, to obtain

$$R^j(s) = \frac{1}{j} \left( \sigma (E_j(y|s) - y(s)) + \sum_{i=1}^j (E_i(\pi|s) + E_{i-1}(v|s)) \right).$$

There is no term premium in this model, though real-interest rate shocks  $v$  do introduce deviations from the usual arbitrage conditions and could be interpreted as time-varying spreads. It is worth remarking that long-term rates in the model are mainly driven by long-term inflation expectations. In the limit, the annualized rate is equal to the unconditional inflation expectation,

$$\lim_{j \rightarrow \infty} R^j(s) = E\pi.$$

Cost-push and real-rate shocks may be persistent, but they are assumed to be stationary: only private-sector beliefs can shift long-run inflation expectations in the model. This observation plays an important role later on regarding the maturity choice for the interest-rate stabilization goal.

---

<sup>36</sup>There are foundations for such a term to be included in a micro-founded social welfare loss function if, for example, there are transaction frictions. See Woodford (2003) and Rudebusch (2006) for a discussion.

The necessary and sufficient first-order condition associated with the monetary authority’s problem is

$$\kappa\pi(s) + \psi y(s) \leq \rho\sigma R^j(s) \tag{14}$$

with strict equality if  $R(s) > -Z$ .<sup>37</sup> The comparison with the first-order condition in the case of inflation targeting (6) is straightforward: The monetary authority is balancing the stabilization of output and inflation—the terms in the left-hand side of equation (14)—with the stabilization of the nominal rate of choice.

Before proceeding to the results, it is worth to discuss how my theory of interest-rate stabilization relates to Taylor rules with interest-rate smoothing, i.e., which include a lagged policy-rate term. Empirical studies typically find the coefficient on the policy-rate lag to be significant, with values close to but below one.<sup>38</sup> There is also a strong theoretical basis for interest-rate smoothing, particularly for “super-inertial” rules featuring a coefficient on the policy-rate lag greater than unity: Rotemberg and Woodford (1999) and Gianonni and Woodford (2002) show how such rules can be designed to implement the optimal policy in simple models; Levin et al. (1999) argue that a first-difference interest-rate rule performs admirably across a wide range of large-scale models; and Benhabib et al. (2003) show how super-inertial rules can prevent non-stationary equilibria. While sharing the goal of reducing interest-rate volatility, my specification does not include any backward-looking term and, as I will show, works at its best when it is as forward looking as possible by targeting a long-term interest rate.<sup>39</sup> The implications for equilibrium multiplicity are also quite different. Interest-rate smoothing in Taylor rules does not preclude multiple steady states, though it typically rules out a host of non-stationary equilibria when they are super-inertial; in contrast, my approach to interest-rate stabilization implements an unique Markov equilibrium but leaves open the possibility of multiple non-stationary equilibrium paths.

It remains a question on whether (14) can be expressed as a recognizable Taylor rule. This is indeed the case with  $j = 1$ , when (14) is isomorphic to a simple Taylor rule with coefficients on inflation and output given by  $\kappa/\rho\sigma$  and  $\psi/\rho\sigma$ , respectively. For any longer maturity  $j > 1$ , the policy rate can be expressed in terms of the future expectations of the policy rate,

$$R(s) \geq \frac{j\kappa}{\rho\sigma}\pi(s) + \frac{j\psi}{\rho\sigma}y(s) - E_t\left(\sum_{i=1}^{j-1} R_{t+i}\right),$$

or, if we choose to substitute for the terms  $E_t R_{t+j}$ , on  $j$ -step expectations of inflation and output gap. The forward-looking terms on the right-hand side are key to anchor

---

<sup>37</sup>The square term in the duration,  $\sqrt{j}$ , ensures that the targeted nominal rate still responds one-to-one with the policy rate at all horizons.

<sup>38</sup>See Judd and Rudebusch (1998) and Levin et al. (1999), among many others.

<sup>39</sup>Note that, if evaluated at  $j = 1$ , (14) is isomorphic to a simple Taylor rule *without interest-rate smoothing*.

inflation expectations. Simple Taylor rules typically include only contemporaneous or backward-looking terms and do not depend on expectations of future policy rates.

## 6.2 Analytic results

The main result in this section is a sufficient condition to anchor private-sector expectations: Proposition 6.1 states that if the weight on interest-rate stabilization is larger than a threshold, then there is a unique Markov equilibrium. Unfortunately, an analytic result is available only for i.i.d. shocks, and I have to resort to numerical methods to explore the case of persistent shocks.

**Proposition 6.1.** *Let  $F(s'|s) = F(s')$  for all  $s, s' \in S$ . If  $\rho > \frac{\kappa^2 + \psi(1-\beta)}{\sigma\kappa}$ , then there is a unique Markov equilibrium.*

*Proof.* In the appendix □

Emphasizing interest-rate stability ensures that the monetary authority does not accommodate any permanent shift in inflation expectations. Say the private sector expects inflation to be below the target, pushing output and future rates to drop below target as well. If the weight on interest-rate stabilization is large enough, the monetary authority will choose to keep nominal interest rates close to the steady state—that is, it will set the policy rate above the ZLB despite output and inflation drifting further away from their targets. By doing so, the private-sector inflation expectations will not be validated, ruling any Markov equilibrium with low inflation.

The threshold in Proposition 6.1 depends on the parameters governing the monetary policy transmission,  $\kappa, \sigma, \beta$ , as well as the relative weight given to output in the policy loss function. Conspicuously absent are any terms regarding the volatility of the shock process or, more surprisingly, the distance to the ZLB,  $Z$ .<sup>40</sup> The reason is that the threshold is not designed to avoid the ZLB at all costs. As the numerical examples below show, the ZLB can be binding in response to shocks.

Unfortunately, there is no guarantee that policy-rate stabilization allows the monetary authority to achieve its output and inflation targets and, more worryingly, that it actually improves welfare upon any of the Markov equilibria that arise under inflation targeting. By construction, the monetary authority’s response to cost-push shocks will be muted and thus further away from optimal. Moreover, now the monetary authority will not fully accommodate real-interest rate shocks, which can typically be stabilized completely if the ZLB is not binding.

Stating the stabilization goal in terms of *long-term* nominal rates ameliorates the costs associated with this framework. The simple reason is that longer rates are mainly

---

<sup>40</sup>The threshold does not depend on the maturity  $j$  because of the normalization I introduced in the monetary authority loss function.

determined by inflation expectations, which is exactly what the policy rate should be reacting strongly in order to rule out the low-inflation Markov equilibrium. The real rate, in contrast, has little weight and thus does not interfere with short- to medium-term stabilization policy.

I should note that Proposition 6.1 proves the uniqueness of Markov equilibria, but it is likely that other rational-expectations equilibria exist. First, history-dependent equilibria are pervasive in repeated-game settings like the present. Second, the interest-stabilization goal leads the nominal interest rate to react less than one-to-one with inflation, so sunspot equilibria can arise as well.

## 7 Numerical results

I complement here the analytic results with a numerical exploration based on standard parameter values in the literature. While the model is admittedly very simple, the numerical exercise provides some further insight on allocations, prices and welfare across Markov equilibria.

### 7.1 A simple calibration

The model is evaluated at the quarterly frequency. I assume log-preferences, so  $\sigma = 1$ , and set the intertemporal discount rate  $\beta$  at .994 such that the annual real interest rate is 2.5 percent. The slope of the NKPC,  $\kappa$ , is .024. This is the value used in Rotemberg and Woodford (1997) and followed countless times for small-scale New Keynesian models. I assume a target inflation of 2 percent annually and set the ZLB accordingly. The weight on the output gap in the social welfare function,  $\lambda$ , is given by the structural parameters, assuming an elasticity of substitution across goods of 5.<sup>41</sup> Regarding the shock processes, I need to inform my choice of the Markov transition matrix  $F(s'|s)$ . For comparison with the literature, I choose to approximate an independent Gaussian auto-regressive process for each shock, with standard deviations  $\sigma_u$  and  $\sigma_v$ , in percentage points, set at .06 and .6, respectively, and auto-correlation coefficients at  $\rho_u = .8$  and  $\rho_v = .9$ .<sup>42</sup> These values roughly approximate the volatility and auto-correlation of the output gap and inflation in the U.S., as given by the CBO output gap and core PCE, when the model is evaluated under discretion, i.e.,  $\psi = \lambda$ , and ignoring the ZLB.



	Optimal policy	Inflation Targeting
<b><i>Social loss</i></b>		
$L_t$	0.023	0.045
<b><i>Inflation and output</i></b>		
$E\pi_t$	2.00	2.00
$Ey_t$	0.00	0.00
$\sigma_y$	1.34	1.35
$\sigma_\pi$	0.287	0.542
$\rho(\pi_t, y_t)$	-0.134	-1
<b><i>Auto-correlations</i></b>		
$\rho(y_t, y_{t-1})$	0.964	0.804
$\rho(\pi_t, \pi_{t-1})$	0.541	0.804

Table 1: Optimal policy and inflation targeting, ignoring the ZLB

## 7.2 Inflation targeting

I start by benchmarking my results against the optimal policy and inflation targeting in an economy without the ZLB. In the first column in Table 1 I compute the optimal policy, from a timeless perspective, as given by the equation  $\pi_t = \lambda/\kappa(y_t - y_{t-1})$ . The first row reports the social welfare loss. All inflation moments are given in annualized percentages and output moments are in percentage deviations from the efficient level (output gap). There is not much to report on the optimal policy except in contrast to inflation targeting, which is reported in the second column. Since there is no ZLB, there is a unique Markov equilibrium. Despite setting the output weight arbitrarily at just half the weight in the social welfare loss function,  $\psi = .5\lambda$ , inflation targeting approximates quite well the optimal policy outcome, with only slightly more output volatility and a more sizable, yet still modest, increase in inflation volatility. Since there is no ZLB, real-rate shocks are perfectly stabilized, and the only variation comes from cost-push shocks, which send inflation and output in opposite directions.

Upon reintroducing the ZLB, I find there are two Markov equilibria, reported in the two columns of Table 2. There is no mistaking which equilibrium is the stabilization or “good” equilibrium. There is indeed little difference between inflation targeting without the ZLB, reported in Table 1, and Equilibrium 1 in Table 2. The ZLB binds only rarely

<sup>41</sup>That is,  $\lambda = \kappa/5 = .005$ . See Woodford (2003) for details on the derivation.

<sup>42</sup>I use the Rouwenhorst method 16 grid points for the state space. See Kopecky and Suen (2010) for a description and discussion of the Rouwenhorst method.

	<b>Inflation Targeting</b>	
	<i>Equilibrium 1</i>	<i>Equilibrium 2</i>
<b><i>Social loss</i></b>		
$L_t$	0.047	0.82
<b><i>Inflation and output</i></b>		
$E\pi_t$	1.96	0.38
$Ey_t$	-0.01	-0.10
$\sigma_y$	1.32	3.07
$\sigma_\pi$	0.569	2.13
$\rho(\pi_t, y_t)$	-0.954	0.746
<b><i>Auto-correlations</i></b>		
$\rho(y_t, y_{t-1})$	0.796	0.822
$\rho(\pi_t, \pi_{t-1})$	0.799	0.857
<b><i>Zero lower bound</i></b>		
$\Pr(R_t = -Z)$	0.0625	0.453
$E\tau_Z$	3.95	11.6

Table 2: Markov equilibria under inflation targeting

and for short spells—about 6 percent of the time and for an average of four quarters, respectively. Indeed, it takes a large real-rate shock for the ZLB to be binding—and even small, positive cost-push shocks can lift the policy rate above the ZLB. It is thus not a great impediment to stabilization policy. In contrast, Equilibrium 2 in the second column of Table 2 features quite large volatility of both output and inflation, as well as a low inflation rate average—though not quite outright deflation. The economy spends close to half of the time at the ZLB, with spells at the ZLB averaging close to three years. Mild real-rate shocks are sufficient to drive the policy rate to the ZLB.

Due to the ZLB, the economy is not linear, and thus the first and second moments reported in Table 2 do not tell the whole story. Figure 1 reports the unconditional distributions for the inflation rate and the output gap in each Markov equilibrium.<sup>43</sup> It is immediately apparent from Figure 1 that both the output gap and the inflation rate distribution are significantly more skewed to the left in the low-inflation equilibrium than in the stabilization equilibrium. Indeed, for the stabilization equilibrium, the distributions for inflation and output are essentially Gaussian, as the ZLB binds only rarely and the variables inherit the underlying distribution for cost-push shocks. However, it is also important to note the large overlap between the distributions across equilibria, particularly for the output gap.

The downside risk in the low-inflation equilibrium is, not surprisingly, associated with the ZLB binding. As inflation expectations are lower, monetary policy has less room to accommodate real rate shocks—in particular, rare but large real rate shocks can drive the real interest rate sharply up in the low-inflation equilibrium.

My numerical analysis is based on the equations described in Section 2 in terms of log-linear deviations from the efficient allocations. Unfortunately, it is computationally complex to solve the underlying non-linear model. I have performed several numerical exercises to validate the use of log-linear approximations, with very positive results. First, I evaluated the non-linear equations using the equilibrium allocations computed with the log-linear equations: I only found very small numerical deviations for a wide range of calibrations. Second, I have also checked whether the monetary authority’s response changes substantially once I drop the quadratic-loss function and use instead the non-linear household preferences: The welfare loss associated with the quadratic approximation was negligible (in consumption equivalent terms).

### 7.2.1 Number of equilibria

My analytic results are silent regarding the exact number of Markov equilibria and it is possible that no equilibrium exists. Here I take up on the determinants of equilibrium existence and complete the discussion with several robustness exercises.

There always two Markov equilibria in a perfect foresight economy. It can also be

---

<sup>43</sup>I simulate the economy for 50,000 periods and use a Kernel density estimator.

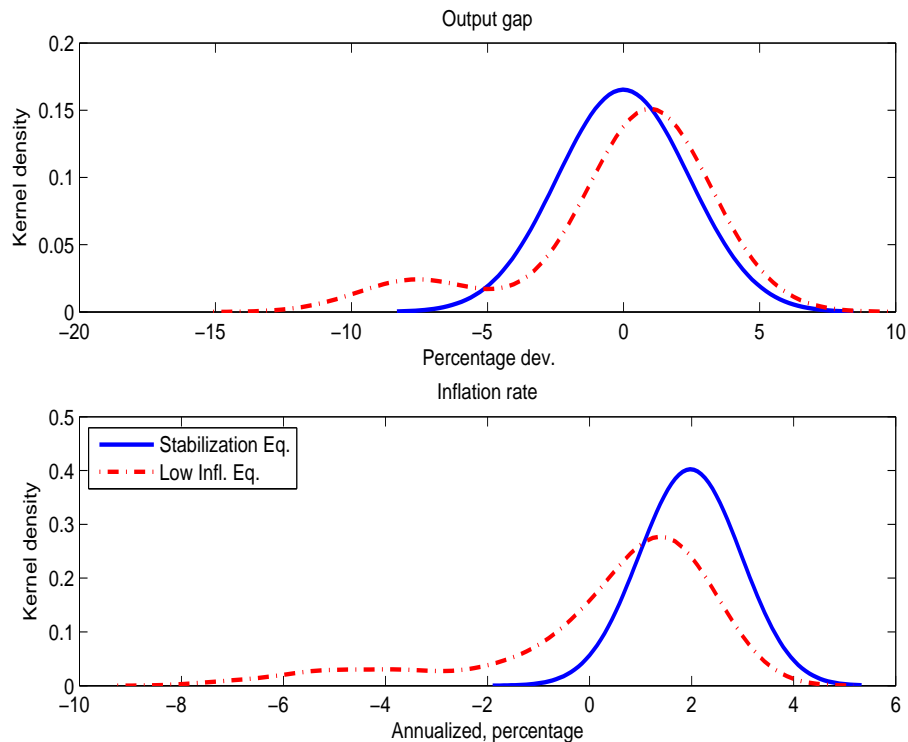


Figure 1: Inflation and output-gap distributions across Markov equilibria

shown that if shocks are sufficiently small, then equilibrium existence is guaranteed in a stochastic economy.<sup>44</sup> Thus the phenomenon of equilibrium non-existence is closely related to large shocks—in particular, as we shall see, to large negative real rate shocks.

Let us attempt an heuristic discussion of the possibility of equilibrium non-existence. The starting observation is that the monetary authority can always accommodate *positive* real rate shocks, independently of their magnitude, but it cannot do so with large *negative* real rate shocks that drive the policy rate to the ZLB. Consider an economy with only i.i.d. real rate shocks, with one of them large enough such that the ZLB is binding. The asymmetric effect of the ZLB implies that, everything else constant, inflation expectations must be revised downwards from their full-stabilization values. Denote the first-step revision to inflation expectations by  $\pi_1^e < 0$  and let us check whether it constitutes a candidate equilibrium.

Once inflation expectations are revised downward, a feedback effect arises via the monetary authority response. For shocks such that, given  $\pi_1^e$ , the ZLB is not binding, the realized inflation will overshoot expectations, but only moderately: Equation (8)

<sup>44</sup>Quite trivially, if shocks are small enough then there is always a stabilization equilibrium with zero probability of hitting the ZLB.

implies that, absent cost-push shocks, actual inflation will lie between expectations and the target, i.e.,  $\pi^e < \pi(s) \leq 0$ . On the other hand, for shocks that drive the policy rate to the ZLB, inflation will be substantially below inflation expectations: From (7), roughly,  $\pi(s) \approx (1 + \nu)\pi_1^e + b < \pi_1^e$ , where  $b < 0$  is a function of parameters and the real-rate shock innovation, which is necessarily negative. It is thus necessary to further revise inflation expectations  $\pi_2^e < \pi_1^e$ . This, in turn, lowers the realization of inflation when the ZLB is binding further and, in addition, may imply that the ZLB is now binding in additional states, bringing into play further downward pressure on inflation expectations. If the downside risk to real rate shocks is large enough, there is then no fixed point for inflation expectations and the economy would slide on a deflationary spiral which, in the strict confines of the linearized equations, does not constitute an equilibrium.<sup>45</sup>

Starting from the baseline economy, I compute the number of equilibria for a wide range of parameters. I start by confirming that large real-rate shocks are behind equilibrium non-existence. Figure 2 displays the number of equilibria for pairs of standard deviations for cost-push shocks—on the horizontal axis—and real-rate shocks—on the vertical axis. The baseline values are indicated by the dot at  $\sigma_u = .06, \sigma_v = .6$ . The non-existence region is squarely in the top of the figure, corresponding to high values of  $\sigma_v$ . The exact threshold does depend on cost-push shocks, as those can also drive the policy rate to the ZLB. That said, the number of equilibria is robustly at two for a wide range of cost-push shock standard deviation values.

Turning to shock persistence, Figure 3 repeats the previous exercise now varying the auto-correlation coefficient of each shock. In order to keep the resulting cost-push and real-rate shock volatility constant, I adjust the standard deviation of the innovations correspondingly. Figure 3 shows there exist at least two Markov equilibria for most of the parameter space: only if both real-rate and cost-push shocks are made very persistent there is no equilibrium.<sup>46</sup>

Both in Figure 2 and 3 there are parameter regions with four or more Markov equilibria. Typically these additional equilibria are very similar to either the low-inflation or the stabilization equilibrium, with only minor changes in the set of states that the ZLB is binding and nearly identical moments for inflation and output.<sup>47</sup> I also explored a wide range of parameters values for  $\kappa, \sigma$  and  $\beta$ , and found that the number of equilibria is robustly just two for all but a few extreme parameter values. Similarly, the full range of

---

<sup>45</sup>In a recent paper, Nakata and Schmidt (2014) derives the conditions for equilibrium existence for two-state Markov shocks in a similar economy.

<sup>46</sup>Are the shocks leading to deflationary spirals plausible? A definitive answer would require a richer model, so I venture here only a conjecture: The levels of volatility and persistence that may trigger non-existence may only be needed if we insist to explain the long sojourn at the ZLB observed in the U.S. with the equilibrium conditions of the stabilization equilibrium, that is, without allowing the possibility of equilibrium dynamics leading to the low-inflation equilibrium.

<sup>47</sup>The reader may wonder how it is that there are four Markov equilibria as the shock volatility tends to zero but only two Markov equilibria in the non-stochastic economy. The reason is that the additional equilibria specify different allocations for the states for whose probability tends to zero.

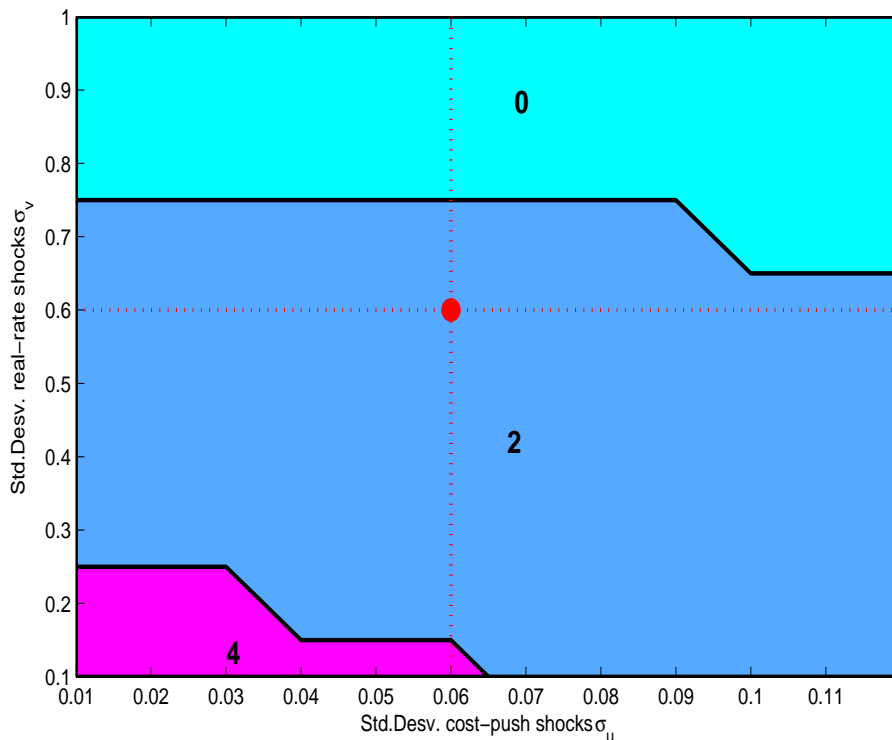


Figure 2: Number of Markov equilibria - Shock volatility

inflation targeting policy parameters,  $\psi \in [0, \lambda]$ , featured exactly two equilibria.

### 7.3 Interest-rate stabilization

I resort again to numerical methods to complete my analysis.<sup>48</sup> First, I ask whether interest-rate stabilization can anchor expectations to a unique equilibrium if shocks are persistent. If so, then I am interested in the costs of doing so, that is, how much the interest-rate stabilization goal disrupts stabilization policy.

Regarding the first question, I find that the threshold presented in Proposition 6.1 remains effective for economies with persistent shocks. I found a unique Markov equilibrium whenever the weight for interest-rate stabilization was set just above the aforementioned threshold, across a wide range of structural and policy parameters. Here the algorithm provided in Section 4 can be easily adapted and again proves very useful, since it allows us to establish equilibrium uniqueness.

Next I explore the equilibrium properties across different rate maturities: 3 months (policy rate) and the five and 10 years (nominal rates). In the first exercise, the weight

<sup>48</sup>The structural parameters are the same as described previously in Section 4.

---

<b><i>Policy parameters</i></b>				
Output weight $\psi$	$\lambda$	$\lambda$	$\lambda$	$.5\lambda$
Maturity $j$	3 months	5 years	10 years	10 years
Rate weight $\rho$	.026	.026	.026	.027
<b><i>Social loss</i></b>				
$L_t$	0.76	0.18	0.12	0.10
<b><i>Inflation and output</i></b>				
$E\pi_t$	3.36	2.66	2.49	2.30
$Ey_t$	0.08	0.04	0.03	0.02
$\sigma_y$	2.13	1.02	0.88	1.25
$\sigma_\pi$	2.25	1.13	0.90	0.89
$\rho(\pi_t, y_t)$	0.848	-0.021	-0.584	-0.29
<b><i>Auto-correlations</i></b>				
$\rho(y_t, y_{t-1})$	0.899	0.857	0.82	0.822
$\rho(\pi_t, \pi_{t-1})$	0.874	0.834	0.818	0.842
<b><i>Zero lower bound</i></b>				
$\Pr(R_t = -Z)$	0.0625	0.0625	0.0625	0.0625
$E\tau_Z$	3.95	3.95	3.95	3.95

---

Table 3: Markov equilibria under nominal rate stabilization, different maturities, and output weights

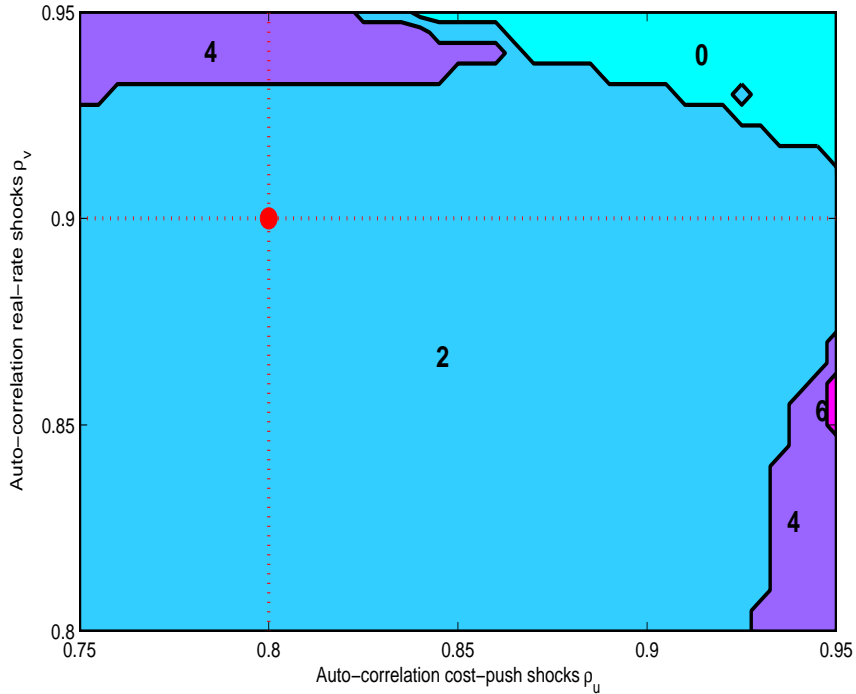


Figure 3: Number of Markov equilibria - Shock persistence

on the output gap in the monetary authority objective function is set equal to its value in the social welfare loss function,  $\psi = \lambda$ . The weight on the interest rate is set just above the threshold implied by Proposition 6.1.

Table 3 documents the results. The first column corresponds to the case of policy-rate stabilization: The results are not very encouraging. The social loss is just below that of the low-inflation equilibrium under inflation targeting. Inflation is way above its target; both output and inflation are very volatile and positively correlated. While we were expecting stabilization policy to fare poorly, the high inflation mean comes as a surprise. The reason is that the monetary authority, once it is given a strong goal for interest-rate stabilization, reacts to inflation-expectations deviations from the target by driving the realized inflation *further* away from the target. Real-rate shocks will always drive realized inflation below expectations when the ZLB is binding. Thus, if expectations are below the target, then  $\pi < E\pi < 0$  whether the ZLB is binding or not, which obviously cannot be an equilibrium. Instead, when expectations are above target, realized inflation is above expectations when the ZLB is not binding, and below expectations when the ZLB is binding.

Fortunately, interest-rate stabilization performs much better when the monetary authority's objective function targets medium- and long-term nominal rates instead of the



policy rate. The second and third columns in Table 3 report the equilibrium properties of a stabilization goal in terms of the 5- and 10-year nominal rates, respectively. In both cases, welfare is substantially higher than it is in the low-inflation equilibrium under inflation targeting. The unconditional mean of inflation remains above target, but only by about half a percentage point. Both inflation and output volatility are back within agreeable values. Indeed, using the 10-year nominal rate approximates quite well the stabilization equilibrium, with output and inflation moving in opposite directions—the telltale sign that monetary policy is properly responding to shocks.

The last column in Table 3 computes the equilibrium that results from combining inflation targeting—arbitrarily setting the output weight to half the social welfare’s, as in Section 4—with a stabilization goal for the 10-year nominal rate. Welfare improves by reducing the inflation mean, but actual stabilization policy worsens. Indeed, the welfare loss increases if the output weight is reduced in the case of the policy rate and the 5-year nominal rate (not reported). It is possible to tweak the policy parameters—both the output and interest-rate weight—to further reduce welfare loss, bringing interest-rate stabilization within earshot of matching the welfare properties of the stabilization equilibrium under inflation targeting.

## 8 Extensions

There is little doubt so far that the experience with inflation targeting has been very successful. Several inflation-targeting central banks have kept their policy rates virtually at zero for extended periods, yet inflation expectations have remained anchored. I briefly ask here whether inflation inertia, fiscal policy or a temporary interest-rate peg can pin down the economy to a single Markov equilibrium and, if so, whether it is desirable to do so.<sup>49</sup>

### 8.1 Inflation inertia

In the simple model presented in Section 2, inflation is persistent in as much as the underlying shocks are. Researchers have sought to include a propagation mechanism for inflation in the model—what is commonly referred as inflation inertia or intrinsic persistence. Here I ask whether inflation inertia can overturn the multiplicity of equilibria under inflation targeting.

Gali and Gertler (1999) assume that a fraction  $\omega$  of those firms able to adjust prices in a given period use a simple rule of thumb, setting their price equal to the average of newly adjusted prices of the last period plus lagged inflation. Christiano et al. (2005) and many others propose instead an alternative specification where prices are automatically adjusted

---

<sup>49</sup>See Bullard (2010) for a similar discussion in the context of U.S. and Japan monetary policy and the results in Benhabib et al. (2001).

according to an exogenous indexation rule,  $p_t(i) = p_{t-1}(i) + \gamma\pi_{t+1}$ . Both approaches lead to a hybrid variant of the NKPC that includes lagged inflation and nests (1) when either  $\omega$  or  $\gamma$  are equal to zero.<sup>50</sup>

Inflation inertia introduces an additional complication for the computation of Markov equilibria, since now lagged inflation is a state variable.<sup>51</sup> The monetary authority's decision has thus an inter-temporal aspect since it internalizes the impact that the policy rate has in present and future inflation and, in turn, future policy rate decisions.

It is still possible to provide some partial results for a non-stochastic economy. First, there are always at least two steady states, given by  $\pi = -Z$  and  $\pi = 0$ . Quite trivially, the options for the monetary authority under inflation inertia remain severely constrained by the ZLB, being only possible to surprise inflation expectations on the downside if the latter are low to start with. Inflation inertia does change the sacrifice ratio but, as both output and inflation are below target, it does not provide any reason for the monetary authority to raise the policy rate when faced with low inflation expectations.

Perhaps a more interesting question is whether an economy that was previously anchored to the stabilization equilibrium,  $\pi_{-1} = 0$ , may transition to the low-inflation equilibrium. I can show that, as long as the inflation inertia is not too large, there exist Markov equilibria such that inflation converges monotonically to the low-inflation steady state. Along the path, the monetary authority acts aggressively and cuts the policy rate all the way to the ZLB. There is, thus, nothing else the central bank can do to arrest the downward path of inflation: The alternative being increasing the policy rate, which would only hasten the transition. Estimates of the hybrid NKPC from Gali and Gertler (1999) and Gali et al. (2005) imply relatively low inflation inertia, within the range of parameter values for which there is a Markov equilibrium that converge monotonically to the ZLB.

The literature using medium- and large-scale models, though, relies instead on price indexation with quite high values for  $\gamma$ , occasionally assuming full price indexation,  $\gamma = 1$ . For such high degrees of inertia, I have to resort to numerical methods. Using the baseline parameters and exploring price-indexation values up to  $\gamma = 1$ , I find there is a Markov equilibrium such that inflation converges to the ZLB, albeit not necessarily monotonically. Price indexation dampens, but does not eliminate, the forward-looking component for inflation. As long as the latter remains, the equilibrium multiplicity appears pervasive and higher  $\gamma$  only slows the transition dynamics. Only at an extreme case, when inflation is *exclusively* backward looking, then there is no transmission from inflation expectations to price setting, and the former have to be necessarily anchored to lagged inflation,  $\pi_{-1}$ .<sup>52</sup>

A different kind of inflation inertia may arise through learning. While Markov equilibria are a narrow subset of all rational expectations equilibria, there is a question of whether

---

<sup>50</sup>Indeed, both specifications are isomorphic though, unfortunately, the equivalence relationship involves parameters other than  $\omega, \gamma$ .

<sup>51</sup>See Appendix for the equilibrium definition under price targeting, which can easily be reformulated in terms of lagged inflation instead of the price level.

<sup>52</sup>Neither specifications actually encompass this case unless  $\beta = 0$ .

all Markov equilibria are stable under some form of learning criterion. If the literature on Benhabib et al. (2001) is any indication, all steady states may have some converging learning dynamics associated.<sup>53</sup> Bullard (2010) also points out that the Japanese experience suggests a liquidity trap is a real possibility. In any case, introducing learning dynamics in the context of policy discretion remains a daunting task beyond this paper's scope.

## 8.2 Fiscal policy

In the model presented in Section 2, monetary policy is active, according to the definition of Leeper (1991). For an equilibrium to exist, the fiscal authority must be passive, that is, it should set taxes such that the inter-temporal government budget constraint is satisfied. An interesting question is whether the fiscal authority may be passive if inflation expectations are centered around the prescribed targets, but turn to an active regime otherwise. If so, the low-inflation Markov equilibrium may be ruled out and inflation expectations anchored to the stabilization equilibrium.

Let  $B_t$  be one-period nominal government debt, that earns interest at the rate  $R_t$ . Nominal balances are denoted by  $M_t$ , the price level by  $P_t$ , and lump-sum taxes, in real terms, by  $\tau_t$ . The government budget constraint is

$$B_t + M_t + P_t\tau_t = P_tg + M_{t-1} + R_{t-1}B_{t-1}, \quad (15)$$

where  $g$  are government expenditures, in real terms, which are assumed to be constant. It is also necessary to specify a money demand equation:

$$\frac{M_t}{P_t} = Y_tg(R_t) \quad (16)$$

where  $g$  is a twice-differentiable, decreasing function with  $g'(1) = 0$ , which mimics a satiation point at the ZLB. Because taxation is lump-sum, government debt is not a payoff-relevant variable and thus there is no need to modify the Markov equilibrium definition from Section 4.

Not surprisingly, if fiscal policy is globally passive, then it accommodates all Markov equilibria in my analysis. A trivial example would be a fiscal policy that balances the budget constraint at every period. A more interesting question is whether standard specifications, under parameter values such that fiscal policy is locally passive, also imply that fiscal policy is globally passive given the non-linearity introduced by the ZLB. Consider the classic formulation in Leeper (1991):

$$\tau_t = \gamma_0 + \gamma \frac{B_{t-1}}{P_{t-1}}.$$

---

<sup>53</sup>See Eusepi (2007) and Evans et al. (2008).

As it is well known, if  $|\gamma - \beta^{-1}| < 1$ , then the fiscal policy is passive in a linear economy and, faced with active monetary policy, there is a unique equilibrium. Does this fiscal rule accommodate multiple Markov equilibria when we consider the ZLB? The answer is clearly yes. The long-run real rate is invariant across both the stabilization and the low-inflation Markov equilibria, at  $r_t = \beta^{-1}$ . Seignorage revenues will be lower in the low-inflation equilibrium, but as long as the speed of fiscal adjustment,  $\gamma$ , is fast enough relative to the real rate, debt will be stabilized.<sup>54</sup> I briefly explored some alternative linear or log-linear specifications—e.g., including a stabilization motive in fiscal policy—with the same result.<sup>55</sup>

From a normative standpoint, it may be desirable to *design* fiscal policy such that it is only passive when inflation expectations are anchored in the stabilization equilibrium. This, for example, is advocated in Eggertsson (2006). It appears far-fetched, though, to propose to tie down irresponsible fiscal behavior to what would appear as a failure of the monetary authority. It is also not clear which institutions would be needed to support a “commitment to be irresponsible.”

Another possibility is that passive fiscal policy has some natural or self-imposed limits. For example, assume there is an upper bound to the amount of lump-sum taxes that can be collected,  $\bar{\tau}$ . A fiscal policy aimed to zero debt growth,  $b_t = b_{t-1}$ , may or may not be passive, depending whether the shocks are large enough such that the upper bound on taxes binds regularly. If so, debt would enter an explosive path and fiscal policy would effectively be passive. It is therefore possible that fiscal policy is passive in the stabilization policy but, due to the lower seignorage revenues and higher inflation and output volatility, active in the low-inflation equilibrium.

A related issue is whether the ZLB is such a constraint on policy as I made it out to be. Several central banks embarked on asset purchase programs as an additional policy tool when the nominal rate hit the ZLB. Were we to find these policies as effective as the policy rate, the central bank’s hands would not be tied in the face of low-inflation expectations. However, the logic of the rational expectations is quite demanding: The effect of the asset purchases must be unbounded to ensure a unique equilibrium. Note that my results did not hinge on the particular value of the lower bound on the policy rate,  $Z$ . We could conceivably set  $Z$  as large as desired, capturing a policy-rate equivalent impact of asset purchases, and we would yet face the same perils.

---

<sup>54</sup>In a stochastic economy, one needs to make sure that debt converges almost surely as the real interest rate will fluctuate over time. This is the case here given the assumption of Markov equilibria and the bounded support for shocks.

<sup>55</sup>Quantitatively, the calibration in Section 4 suggests that the drop in seignorage is relatively modest, as inflation remains positive and output does not drift far from steady state.

### 8.3 Commitment to keep interest rates low

As we saw in Section 4, the low-inflation equilibrium is also characterized by output being below its efficient level and rates being abnormally low. However, in a stochastic economy, the policy rate will occasionally rise above the ZLB: In the numerical exercise in Section 7, the policy rate is actually above the ZLB about half of the time. Here I explore whether the monetary authority could dispel the low inflation expectations by committing to keeping the policy rate low for a prolonged time. This is exactly the prescription given by Eggertsson and Woodford (2003) in response to a temporary real interest rate shock and, arguably, tracks closely some of the communications of the Federal Reserve—what is often if loosely referred as “forward guidance.”

To capture the idea of a commitment to keep the policy rate low, I assume that the monetary authority commits to a temporary interest-rate peg at the ZLB before switching back to, say, inflation targeting.<sup>56</sup> The duration of the interest-rate peg can be “time-dependent,” i.e., for a fixed number of periods; or be made state contingent; or a combination of both, e.g., committing to keep rates at the ZLB for, say, three periods after the output is above a certain threshold.

I will next show that, for a very general specification of the interest-rate peg and exit process, there are equilibrium dynamics leading to each of the Markov equilibria under inflation targeting. The resulting inflation and output paths will be different depending on which Markov equilibrium prevails once the peg ends. The inherent multiplicity problem is thus not resolved, and it actually extends to the transition dynamics during the peg stage.

Formally, let  $T$  be the exit date from the peg—possibly a random variable if the exit is made state contingent. Assume, for simplicity, that under inflation targeting there are two Markov equilibria, with inflation levels  $\pi^*, \pi^{**} : S \rightarrow \mathfrak{R}$ . Next I proceed to solve for output and inflation backwards from date  $T$  for each Markov equilibrium simply using the NKPC (1) and the Euler equation (2) with the policy rate at the ZLB. At date  $T - 1$ , for a given state  $s_{T-1}$ , the Euler equation (2) pins down output

$$\sigma y_{T-1}^*(s_{T-1}) = Z + \sigma E_1(y^*|s_{T-1}) + E_1(\pi^*|s_{T-1}) + \nu(S_{t-1})$$

where I have assumed that Markov equilibrium  $\pi^*$  will be in place at date  $T$ .<sup>57</sup> The NKPC (1) determines inflation as well

$$\pi_{T-1}^*(s_{T-1}) = \kappa y_{T-1}^*(s_{T-1}) + \beta E_1(\pi^*|s_{T-1}) + u(S_{t-1}).$$

We thus obtain schedules  $\pi_{T-1}^*, y_{T-1}^* : S \rightarrow \mathfrak{R}$ —that is, an inflation rate and output gap for each possible realization of the state at date  $T - 1$  which, by construction, satisfy the

<sup>56</sup>Another possibility is to adopt the suggestion in Eggertsson and Woodford (2003) of a price-level target: As shown in Section 5, a price-level target does not solve the problem of multiple equilibria.

<sup>57</sup>The notation for expectations is the one used previously. If the exit date is a random variable, one needs to formally extend the expectation operator to be measurable with respect to exit dates—doing so, while cumbersome, poses no difficulty.

equilibrium equations. The date  $T - 1$  schedules, in turn, can be used in identical manner to solve for date  $T - 2$  schedules, and so on, until we obtain a complete path—that is, a collection of schedules for dates  $t = 0, \dots, T$ —leading to the Markov equilibrium  $\pi^*$ .

Similarly, we can solve for the path leading to the other Markov equilibrium  $\pi^{**}$ , obtaining a collection of schedules  $\pi_t^{**}, y_t^{**} : S \rightarrow \mathfrak{R}$  for dates  $t = 0, \dots, T$ . Both transition paths will be equilibrium objects and, generically, will be distinct from each other.

It may be worth remarking the differences with Eggertsson and Woodford (2003). Their analysis concerns only a temporary real-rate shock: No consideration is given to other equilibria but the one centered around stabilization. In their context, the commitment to keep the rate low relieves the downward pressure on the real rate by rising inflation expectations. This is, though, ineffective if the reason why the real rate is lower is a downward shift in inflation expectations in the first place. Indeed, in order to rule out the low inflation equilibrium, it is necessary for the monetary authority to disavow the private-sector beliefs—which unambiguously requires to set the rate *higher* than expected.

## 9 Conclusions

The results in this paper should be, first and foremost, a word of caution about formulating monetary policy in terms of nominal targets. The central bank may be committed, and understood to be so beyond any doubt, to pursue the specified targets and yet find itself unable to achieve its goals if private-sector expectations do not comply. Researchers and policymakers alike should be aware of the additional equilibria and explore possible designs that ensure that the desired equilibrium is uniquely implemented.

## References

- Adam, Klaus and Roberto M. Billi**, “Optimal Monetary Policy Under Commitment with a Zero Bound on Nominal Interest Rates,” *Journal of Money, Credit and Banking*, 2006, 38 (7), 1877–1905.
- **and** —, “Discretionary monetary policy and the zero lower bound on nominal interest rates,” *Journal of Monetary Economics*, 2007, 54, 728–752.
- Albanesi, Stefania, V.V. Chari, and Lawrence J. Christiano**, “Expectation Traps and Monetary Policy,” *Review of Economic Studies*, October 2003, 70(4), 715–742.
- Armenter, Roc**, “A General Theory (and Some Evidence) of Expectation Traps in Monetary Policy,” *Journal of Money, Credit and Banking*, 2008, 40 (5), 867–895.
- **and Martin Bodenstein**, “Can the U.S. Monetary Policy Fall (Again) in an Expectation Trap?,” *Macroeconomics Dynamics*, 2008, 12 (5), 664–693.

- Barro, Robert J. and D.B. Gordon**, “A Positive Theory of Monetary Policy in a Natural Rate Model,” *Journal of Political Economy*, 1983, 91 (4), 589–610.
- Bassetto, Marco**, “Equilibrium and government commitment,” *Journal of Economic Theory*, 2005, 124 (1), 79 – 105.
- Benhabib, Jess, Stephanie Schmitt-Grohe, and Martin Uribe**, “The Perils of Taylor Rules,” *Journal of Economic Theory*, January 2001, 96, 40–69.
- , **Stephanie Schmitt-Grohé, and Martín Uribe**, “Backward-looking interest-rate rules, interest-rate smoothing, and macroeconomic instability,” Technical Report, National Bureau of Economic Research 2003.
- Benigno, Pierpaolo and Michael Woodford**, “Optimal Monetary and Fiscal Policy: A Linear-Quadratic Approach,” in “NBER Macroeconomics Annual 2003, Volume 18” NBER Chapters, National Bureau of Economic Research, Inc, December 2004, pp. 271–364.
- and – , “Inflation Stabilization And Welfare: The Case Of A Distorted Steady State,” *Journal of the European Economic Association*, December 2005, 3 (6), 1185–1236.
- Berg, Claes and Lars Jonung**, “Pioneering Price Level Targeting: The Swedish Experience 1931-1937,” *Journal of Monetary Economics*, 1999, 43, 525–551.
- Bernanke, Ben S.**, “A Century of U.S. Central Banking: Goals, Frameworks, Accountability,” July 2013. Speech, National Bureau of Economic Research.
- and **Frederic S. Mishkin**, “Inflation Targeting: A New Framework for Monetary Policy?,” *Journal of Economic Perspectives*, 1997, 11 (2), 97–116.
- and **Michael Woodford**, eds, *The Inflation-Targeting Debate* National Bureau of Economic Research University of Chicago Press Chicago 2005.
- Billi, Roberto M.**, “Optimal Inflation for the US Economy,” *American Economic Journal: Macroeconomics*, 2011, 3 (3), 29–52.
- Boragan, Cuba-Borda Pablo Aruoba and Frank Schorfheide**, “Macroeconomic Dynamics near the ZLB: A Tale of Two Equilibria,” June 2014. Manuscript.
- Bordo, Michael and Pierre Siklos**, “Central bank credibility, reputation and inflation targeting in historical perspective,” Working Paper 20693, NBER November 2014.
- Braun, Anton R., Lena Mareen Korber, and Yuchiro Waki**, “Some Unpleasant Properties of Log-Linearized Solutions When the Nominal Rate Is Zero,” September 2012. Federal Reserve Bank of Atlanta, working paper 2012-5.

- Bullard, James**, “Seven Faces of the Peril,” *Federal Reserve Bank of St. Louis Review*, September/October 2010, *92* (5), 339–352.
- Carney, Mark**, “Writing the path back to target,” March 2015. Speech, University of Sheffield.
- Chari, V.V., Lawrence J. Christiano, and Martin Eichenbaum**, “Expectation Traps and Discretion,” *Journal of Economic Theory*, August 1998, *81* (2), 462–492.
- Christiano, Lawrence J., Martin Eichenbaum, and Charles L. Evans**, “Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy,” *Journal of Political Economy*, February 2005, *113* (1), 1–45.
- Clarida, Richard, Jordi Gali, and Mark Gertler**, “The Science of Monetary Policy: A New Keynesian Perspective,” *Journal of Economic Literature*, December 1999, *37* (4), 1661–1707.
- Cochrane, John H.**, “The New-Keynesian Liquidity Trap,” January 2015. Manuscript, University of Chicago Booth School of Business.
- Dotsey, Michael and Andreas Hornstein**, “On the Implementation of Markov-Perfect Monetary Policy,” July 2011. Federal Reserve Bank of Philadelphia, working paper 11-29.
- Draghi, Mario**, “Monetary policy in a prolonged period of low inflation,” May 2014. Speech, ECB Forum on Central Banking.
- Eggertsson, Gauti**, “The Deflation Bias and Committing to Being Irresponsible,” *Journal of Money, Credit and Banking*, March 2006, *38* (2), 283–321.
- and **Michael Woodford**, “The Zero Bound on Interest Rates and Optimal Monetary Policy,” *Brookings Papers on Economic Activity*, 2003, *1*, 139–233.
- Eusepi, Stefano**, “Learnability and Monetary Policy: A Global Perspective,” *Journal of Monetary Economics*, 2007, *54* (4), 1115–1131.
- Evans, George, Eran Guse, and Seppo Honkapohja**, “Liquidity Traps, Learning, and Stagnation,” *European Economic Review*, 2008, *52* (8), 1438–1463.
- Fernandez-Villaverde, Jesus, Grey Gordon, Pablo Guerron-Quintana, and Juan F. Rubio-Ramirez**, “Nonlinear Adventures at the Zero Lower Bound,” 2012. Federal Reserve Bank of Philadelphia, Working paper 12-10.
- Gali, Jordi**, *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Applications*, Princeton University Press, 2015.



- Gali, Jordi and M. Gertler**, “Inflation Dynamics: A structural Econometric Approach,” *Journal of Monetary Economics*, 1999, 44 (2), 195–222.
- , **Mark Gertler, and J. David Lopez-Salido**, “Robustness of the estimates of the hybrid New Keynesian Phillips curve,” *Journal of Monetary Economics*, September 2005, 52 (6), 1107–1118.
- Gianonni, M and M Woodford**, “Optimal Interest-Rate Rules: II. Applications”, NBER Working Paper n° 9420,” 2002.
- Judd, John P. and Glenn D. Rudebusch**, “Taylor’s rule and the Fed, 1970-1997,” *Economic Review*, 1998, pp. 3–16.
- Jung, T., Y. Teranishi, and T. Watanabe**, “Optimal Monetary Policy at the Zero-Interest-Rate Bound,” *Journal of Money, Credit and Banking*, 2005, 29, 97–133.
- King, Robert G. and Alexander L. Wolman**, “Monetary Discretion, Pricing Complementarity and Dynamic Multiple Equilibria,” *Quarterly Journal of Economics*, 2004, 119 (4), 1513–1553.
- Kopecky, Karen A. and Richard M.H. Suen**, “Finite State Markov-Chain Approximations to Highly Persistent Processes,” *Review of Economic Dynamics*, 2010, 13, 701–714.
- Leeper, Eric M.**, “Equilibria under ‘active’ and ‘passive’ monetary and fiscal policies,” *Journal of Monetary Economics*, February 1991, 27 (1), 129–147.
- Levin, Andrew T, Volker Wieland, and John Williams**, “Robustness of simple monetary policy rules under model uncertainty,” in “Monetary policy rules,” University of Chicago Press, 1999, pp. 263–318.
- Lopez-Salido, David, Andrew Levin, Edward Nelson, and Tack Yun**, “Limitations of the Effectiveness of Forward Guidance at the Zero Lower Bound,” *International Journal of Central Banking*, 2010, 6 (1), 143–189.
- Mertens, K. and Morten Ravn**, “Fiscal Policy in an Expectations Driven Liquidity Trap,” April 2014. forthcoming, *Review of Economic Studies*.
- Nakata, Taisuke and Sebastian Schmidt**, “Conservatism and Liquidity Traps,” Finance and Economics Discussion Series 2014-105, Board of Governors of the Federal Reserve System 2014.
- Nakov, Anton**, “Optimal and Simple Monetary Policy Rules with Zero Floor on the Nominal Interest Rate,” *International Journal of Central Banking*, June 2008, 4 (2), 73–127.

- Piazza, Roberto**, “Self-fulfilling Deflations,” October 2015. Manuscript, Bank of Italy.
- Rogoff, Kenneth**, “The Optimal Degree of Commitment to an Intermediate Monetary Target,” *Quarterly Journal of Economics*, 1985, *100*, 1169–1189.
- Rotemberg, Julio J. and Michael Woodford**, “An Optimization-Based Econometric Framework for the Evaluation of Monetary Policy,” *NBER Macroeconomics Annual*, 1997, *12*, 297–346.
- Rotemberg, Julio J and Michael Woodford**, “Interest rate rules in an estimated sticky price model,” in “Monetary policy rules,” University of Chicago Press, 1999, pp. 57–126.
- Rudebusch, Glenn D.**, “Monetary Policy Inertia: Fact or Fiction?,” *International Journal of Central Banking*, December 2006, *2* (4).
- Schmidt-Grohe, S. and Martin Uribe**, “The Making of a Great Contraction with a Liquidity Trap and a Jobless Recovery,” December 2013. Manuscript, Columbia University.
- Siu, Henry E.**, “Time Consistent Monetary Policy with Endogenous Price Rigidity,” *Journal of Economic Theory*, 2008, *138* (1), 184–210.
- Svensson, Lars**, “Price Level Targeting vs. Inflation Targeting,” *Journal of Money, Credit and Banking*, 1999, *31* (31), 277–295.
- Svensson, Lars E. O.**, “Optimal Inflation Targets, “Conservative” Central Banks, and Linear Inflation Contracts,” *American Economic Review*, March 1997, *87* (1), 98–114.
- , “Inflation targeting: Should It Be Modeled as an Instrument Rule or a Targeting Rule?,” *European Economic Review*, May 2002, *46* (4-5), 771–780.
- Vestin, David**, “Price-Level Versus Inflation Targeting,” *Journal of Monetary Economics*, 2006, *53* (7), 1361–1376.
- Walsh, Carl E.**, “Optimal Contracts for Central Bankers,” *American Economic Review*, March 1995, *85* (1), 150–167.
- Wolman, Alex**, “Real Implications of the Zero Bound on Nominal Interest Rates,” *Journal of Money, Credit and Banking*, 2005, *37* (2), 274–296.
- Woodford, Michael**, “The Taylor Rule and Optimal Monetary Policy,” January 2001. Manuscript, Columbia University.
- , *Interest and Prices*, Princeton: Princeton University Press, 2003.

**Yellen, Janet L.**, “Normalizing Monetary Policy: Prospects and Perspectives,” March 2015. Speech at the “The New Normal Monetary Policy” conference, Federal Reserve Bank of San Francisco.

# A Appendix

## A.1 Section 4

*Proof of Proposition 4.1.* Assume economy  $\xi$  has one equilibrium. I will show that, generically, there is at least another equilibrium.

Consider the vector-valued function  $\Gamma : \mathfrak{R}^n \rightarrow \mathfrak{R}^n$  given by

$$\Gamma_i(g; \xi) = \min \{ \pi_i^u(g; \xi), \pi_i^b(g; \xi) \} \quad (17)$$

for each  $i = 1, 2, \dots, n$ , where, in a slight abuse of notation,

$$\begin{aligned} \pi_i^b(g; \xi) &= (1 + \nu + \beta) E_1(g|s_i) - \beta E_2(g|s_i) + A(s_i), \\ \pi_i^u(g; \xi) &= \frac{\psi}{\kappa^2 + \psi} (\beta E_1(g|s_i) + u(s_i)). \end{aligned}$$

Each vector-valued function can be written concisely in matrix form as

$$\begin{aligned} \pi^b(g; \xi) &= C^b g + A, \\ \pi^u(g; \xi) &= C^u g + u. \end{aligned}$$

Clearly  $g$  is an equilibrium if and only if  $\Gamma(g; \xi) = g$ . Let  $g^*$  be an equilibrium whose existence is the premise of the proof. Let  $h(g; \xi) = \Gamma(g; \xi) - g$  and

$$H = \{g \in \mathfrak{R}^n : h(g; \xi) \geq 0\}.$$

The set  $H$  is nonempty, since it includes  $g^*$ . It is also closed, since  $h$  is continuous. It is also convex, since  $\Gamma$  is the minimum of an ensemble of linear functions and thus weakly concave. In the next two lemmas I prove that  $H$  has a nonempty interior in a generic economy and is bounded.

**Lemma A.1.** *The set  $H$  has, generically, a nonempty interior.*

*Proof.* Let  $e \in \{0, 1\}^n$  with  $e_i = 1$  if  $\pi_i^b(g^*; \xi) \leq \pi_i^u(g^*; \xi)$ ,  $e_i = 0$  otherwise, for all  $i = 1, 2, \dots, n$ . Since  $\Gamma(g^*; \xi) = g^*$ , the equilibrium satisfies

$$e \circ (C^b g^* + A) + (1 - e) \circ (C^u g^* + u) = g^*. \quad (18)$$

Generically,  $\pi_i^b(g^*; \xi) < \pi_i^u(g^*; \xi)$  for all  $i$  such that  $e_i = 1$ : otherwise, adding  $\pi_i^b(g^*; \xi) = g_i$  to (18) delivers a system of  $n+1$  linear equations and  $n$  unknowns, which has generically no solution. Thus, there exists  $\delta > 0$  such that for all  $g \in B_\delta(g^*) = \{g \in \mathfrak{R}^n : \|g - g^*\| < \delta\}$ ,  $\Gamma(g; \xi) = e \circ (C^b g + A) + (1 - e) \circ (C^u g + u)$ . Thus,  $h$  is linear in  $B_\delta(g^*)$  with  $D\Gamma(g; \xi) = e \circ C^b + (1 - e) \circ C^u$ . Generically  $D\Gamma(g^*; \xi)$  has an inverse and thus for any  $\delta > 0$  there exists  $x \in \mathfrak{R}^n$ ,  $\|x\| < \delta$  such that  $D\Gamma(g^*; \xi)x > \mathbf{1}$ , and hence  $h(g + x; \xi) > 0$ . Hence, the interior of  $H$  is nonempty  $\square$

**Lemma A.2.** *The set  $H$  is, generically, bounded.*

*Proof.* By Lemma A.1 there exists, generically, a point  $g'$  such that  $h(g'; \xi) > 0$ . Let  $X \in \mathfrak{R}^n$  be the set of normalized vectors  $x \in \mathfrak{R}^n$  of length 1. Any point  $g \in \mathfrak{R}^n$  can be expressed as  $g = g' + x\delta$  for some  $x \in X$  and scalar  $\delta \geq 0$ .

Consider first the set  $X^+ = \{x \in X : \sum x_i \geq 0\}$ . Note that

$$h(g' + x\delta; \xi) \leq \pi^u(g'; \xi) - g' - \delta \left( x - \frac{\psi\beta}{\kappa^2 + \psi} Fx \right)$$

where  $F$  is the matrix of transition probabilities. Let  $\bar{x} = \max\{x_i \in x\}$ . Clearly,  $\bar{x} \geq Fx$  and thus

$$\bar{x} - \frac{\psi\beta}{\kappa^2 + \psi} Fx \geq \bar{x} \left( 1 - \frac{\psi\beta}{\kappa^2 + \psi} \right)$$

Since  $x \in X^+$ ,  $\bar{x} \geq n^{-\frac{1}{2}} > 0$ . There exists a finite  $\delta > 0$  such that

$$\max_i \{\pi_i^u(g'; \xi) - g'_i\} - \delta n^{-\frac{1}{2}} \left( 1 - \frac{\psi\beta}{\kappa^2 + \psi} \right) < 0.$$

Thus for any  $x \in X^+$ ,  $g' + x\delta \notin H$  since for at least one component  $i$  (namely, any  $i$  such that  $x_i = \bar{x}$ ), we have that

$$\begin{aligned} h(g' + x\delta; \xi) &\leq \pi_i^u(g'; \xi) - g'_i - \delta \left( x_i - \frac{\psi\beta}{\kappa^2 + \psi} Fx \right), \\ &\leq \pi_i^u(g'; \xi) - g'_i - \delta \bar{x} \left( 1 - \frac{\psi\beta}{\kappa^2 + \psi} \right), \\ &\leq \max_i \{\pi_i^u(g'; \xi) - g'_i\} - \delta n^{-\frac{1}{2}} \left( 1 - \frac{\psi\beta}{\kappa^2 + \psi} \right) < 0. \end{aligned}$$

Consider now the complement set to  $X^+$ . By continuity of  $h$ , there exists  $\varepsilon > 0$  such that for all  $x \in X$ ,  $\sum x_i > -\varepsilon$ , then  $g' + x\delta \notin H$ . Let,  $X^- = \{x \in X : \sum x_i \leq -\varepsilon\}$ . Note that for any  $g \in \mathfrak{R}^n$  and scalar  $\tau$ ,

$$h(g - \tau; \xi) \leq \pi^b(g - \tau; \xi) - (g - \tau) = \pi^b(g; \xi) - g - \nu\tau.$$

Consider the hyperplane  $Z = \{g \in \mathfrak{R}^n : \sum g_i = \sum g'_i\}$ . By construction,  $g' \in Z$  so  $Z \cap H$  is nonempty. In addition,  $Z \cap H$  is also compact since any  $g \in Z \cap H$  can be expressed as  $g = g' + x\|g - g'\|$  for some  $x \in X^+$ . Thus, each component of  $\pi^b - g$  achieves a maximum in  $Z \cap H$  and

$$\tau = \frac{1}{\nu} \min_i \max_{g \in Z \cap H} \{\pi_i^b(g; \xi) - g_i\}$$

is well defined. Then for any  $\varepsilon > 0$ ,  $g \in \{Z - (\tau + \varepsilon)\}$  implies  $g \notin H$  as  $\pi_j^b(g; \xi) - g_j < 0$  for some  $j$ . For any  $x \in X^-$ , there exists a  $\theta(x) \geq 0$  such that  $g' + \theta(x)x \in \{Z - (\tau + \varepsilon)\}$ . Clearly  $\theta(x)$  is continuous and thus achieves a maximum  $\bar{\theta}$  in  $X^-$ .

Therefore  $H$  is enclosed in a ball of radius  $\max\{\delta, \bar{\theta}\}$  centered at  $g'$  □

The set  $H$  is the intersection of the collection of the  $2n$  half-spaces given by the  $n$  equilibrium equations, each evaluated with the ZLB or not. Let  $K$  be an index set over  $\{1, 2, \dots, n\} \times \{u, b\}$  with functions  $i : K \rightarrow \{1, 2, \dots, n\}$  denoting the state and  $j : K \rightarrow \{u, b\}$  whether the ZLB binds or not. Define the half-space  $H_k$ ,

$$H_k = \left\{ g \in \mathfrak{R}^n : \pi_{i(k)}^{j(k)}(g) - g \geq 0 \right\}.$$

Formally,  $H = \bigcap_K H_k$  and thus  $H$  is a polytope. By the previous two Lemmas,  $H$  is also nonempty and compact, and therefore  $H$  is a solid (or proper) polytope in  $\mathfrak{R}^n$ . Every equilibrium is a vertex, yet not every vertex is an equilibrium: There are “kinks” defined by the intersection of the boundary of the two half-spaces of the same equilibrium equation. The proof proceeds by induction, eventually ruling out the possibility that all vertexes but one are kinks.

For any  $X \subset \mathfrak{R}^n$ , let  $\Upsilon(X) = \{k \in K : \partial H_k \cap X \neq \emptyset\}$ . A point  $g \in \mathfrak{R}^n$  is an equilibrium if and only if  $\Upsilon(g)$  has  $n$  elements and for any two distinct elements  $k, k' \in \Upsilon(g)$ ,  $i(k) \neq i(k')$ .

Consider first the case in which  $\Upsilon(H)$  has  $2n$  elements. Generically, this implies  $H$  has  $2n$   $(n - 1)$ -facets. Let  $A$  be the convex hull from  $\{H_i \cap \partial H : i \in \Upsilon(g^*)\}$ . There exists a point  $g \in \text{int}(H)$ ,  $g \notin A$ —otherwise,  $H = A$  and  $\Upsilon(H)$  would have only  $n + 1$  elements. By the separating hyperplane theorem, there exists a half-space  $C$  such that  $g \in C$ ,  $C \cap A = \emptyset$ . The intersection  $C \cap H$  is a polytope and thus contains at least one vertex  $\tilde{g} \notin \partial C$ . By construction,  $\Upsilon(C \cap H) \cap \Upsilon(g^*) = \emptyset$ ; that is, none of the  $(n - 1)$ -facets of  $C \cap H$  belong to the supporting hyperplane of the  $(n - 1)$ -facets of  $A$ . Thus,  $\Upsilon(\tilde{g})$  is equal to  $K/\Upsilon(g^*)$  and is an equilibrium. Hence,  $g^*$  is not a unique equilibrium.

Consider the case that  $\Upsilon(H)$  has strictly less than  $2n$  elements. Then there exists  $k \in \Upsilon(H)$  such that for any  $k' \neq k$ ,  $k' \in \Upsilon(H)$ ,  $i(k) \neq i(k')$ . Since  $H$  is a proper polytope in  $\mathfrak{R}^n$ , then  $H_k \cap \partial H$  is a proper polytope in  $\mathfrak{R}^{n-1}$  and obviously  $g^* \in H_k$ . By considering the subspace containing  $H_k \cap \partial H$ , the previous steps can be repeated, substituting  $n$  by  $m = n - 1$  and considering only the remaining half-spaces with  $i \neq i(k)$ . Then either there exists another equilibrium or proceed with  $m = n - 2$  and so on. If  $m = 1$ , then there are two vertexes that are both an equilibrium

□

*Proof of Proposition 4.2.* Clearly  $E_1(\pi|s) = E_2(\pi|s) = E\pi$ . Equation (7) can then be simplified to

$$\pi^b(s) = (1 + \nu) E\pi + A(s).$$

Define the function  $\Gamma(x)$  as

$$\Gamma(x) = \sum_s F(s) \min \left\{ \frac{\psi}{\kappa^2 + \psi} (\beta x + u), (1 + \nu) x + A(s) \right\}.$$

$E\pi = x^*$  characterizes a Markov equilibrium, with

$$\pi(s) = \min \left\{ \frac{\psi}{\kappa^2 + \psi} (\beta x^* + u), (1 + \nu) x^* + A(s) \right\}$$

for all  $s \in S$ , if and only if it is a fixed point of  $\Gamma(x^*) = x^*$ . It is straightforward to show that  $g(x) = \Gamma(x) - x$  is weakly concave.

Let  $\alpha_0 = -\nu^{-1} \sum_S F(s) A(s)$ . For  $x < \alpha_0$ ,  $g(x) < 0$  since  $\Gamma(x) \leq \sum_S F(s) ((1 + \nu)x + A(s))$ . Let  $\alpha_1 = \left(1 - \frac{\psi\beta}{\kappa^2 + \psi}\right)^{-1} \frac{\psi}{\kappa^2 + \psi} \sum_S F(s) u$ . For  $x > \alpha_1$ ,  $g(x) < 0$  since  $\Gamma(x) \leq \sum_S F(s) \frac{\psi}{\kappa^2 + \psi} (\beta x + u)$ . Thus, generically, function  $g(x)$  has either no or two solutions.  $\square$

*Proof of Proposition 4.3.* First I show that  $E\pi^* \leq 0$ . By  $\pi(s) \leq \pi^u(s)$  for all  $s \in S$ ,

$$E\pi^* = f'\pi^* \leq \frac{\psi\beta}{\kappa^2 + \psi} E\pi^*$$

since  $f'F = f'$  and  $Eu = 0$ . If  $E\pi^* > 0$  then

$$\frac{\psi\beta}{\kappa^2 + \psi} E\pi^* < E\pi^*$$

and the contradiction  $E\pi^* < E\pi^*$  is obtained. By (1),  $Ey^* = (1 - \beta)/\kappa E\pi^*$  and  $Ey^* \leq 0$  follows from  $\beta < 1$ . Similarly, (2) implies

$$ER^* = E\pi^*.$$

For the second part, by 4.1, there is, generically, at least another distinct Markov equilibrium,  $\{\pi^{**}, y^{**}, R^{**}\}$ . Since  $\pi^* \neq \pi^{**}$ , then generically  $E\pi^* \neq E\pi^{**}$  and hence for at least one of the Markov equilibria  $E\pi^* < 0$ . For that Markov equilibrium,  $Ey^* < 0$  and  $ER^* < 0$  follow from (1) and (2)  $\square$

*Proof of Corollary 4.1.* By (1),  $Ey^* = (1 - \beta)/\kappa E\pi^*$  and the ranking of output follows from  $\beta < 1$ . Similarly, (2) implies

$$ER^* = E\pi^*.$$

$\square$

*Proof of Proposition 4.4.* For any  $s \in S$ ,

$$\begin{aligned} \pi^*(s) &= \min \left\{ \frac{\psi}{\kappa^2 + \psi} (\beta E\pi^* + u), (1 + \nu) E\pi^* + A(s) \right\} \\ &< \min \left\{ \frac{\psi}{\kappa^2 + \psi} (\beta E\pi^{**} + u), (1 + \nu) E\pi^{**} + A(s) \right\} \\ &= \pi^{**}(s) \end{aligned}$$

$\square$

## A.2 Section 5

**Definition 2.** A *Markov equilibrium* given  $\psi$  consists of:

- A policy function  $R(p) : \mathfrak{R} \rightarrow \mathfrak{R}$ ,
- A value function  $V(p) : \mathfrak{R} \rightarrow \mathfrak{R}$ ,
- Allocation functions  $f_\pi(p), f_y(p), f_p(p) : \mathfrak{R} \rightarrow \mathfrak{R}$ , and
- Private-sector expectations functions  $g_\pi(p), g_y(p) : \mathfrak{R} \rightarrow \mathfrak{R}$ ,

such that for all  $p_{-1} \in \mathfrak{R}$ ,

- Policy and value  $R(p_{-1}), V(p_{-1})$  solve

$$V(p_{-1}) = \max_{R \geq -Z} -p^2 - \psi y^2 + \beta V(p)$$

subject to

$$\begin{aligned} p &= p_{-1} + \pi, \\ \pi &= \kappa y + \beta g_\pi(p), \\ R &= \sigma(g_y(p) - y) + g_\pi(p). \end{aligned}$$

- Allocation functions  $f_\pi(p_{-1}), f_y(p_{-1}), f_p(p_{-1})$  satisfy the equilibrium conditions,

$$\begin{aligned} f_p(p_{-1}) &= p_{-1} + f_\pi(p_{-1}), \\ f_\pi(p_{-1}) &= \kappa f_y(p_{-1}) + \beta g_\pi(f_p(p_{-1})), \\ R(p_{-1}) &= \sigma(g_y(f_p(p_{-1})) - f_y(p_{-1})) + g_\pi(f_p(p_{-1})). \end{aligned}$$

- Private-sector expectations  $g_\pi(p_{-1}), g_y(p_{-1})$  satisfy the rational expectations hypothesis,

$$\begin{aligned} g_\pi(p_{-1}) &= f_\pi(p_{-1}), \\ g_y(p_{-1}) &= f_y(p_{-1}). \end{aligned}$$

*Proof of Proposition 5.1.* We start with a partial conjecture regarding the equilibrium functions. For some  $p^* > 0$ , let

$$\begin{aligned} R(p_{-1}) &= -Z \\ f^p(p_{-1}) &= p_{-1} - Z \\ f^\pi(p_{-1}) &= -Z \\ f^y(p_{-1}) &= -(1 - \beta)Z\kappa^{-1} \end{aligned}$$



for all  $p_{-1} \leq p^*$ . For the conjectured law of motion of the price level, clearly if  $p_{-1} \leq p^*$  then  $f^p(p_{-1}) < p_{-1} \leq p^*$ . Thus, the set  $\{p \leq p^*\}$  is absorbing. Since the price level is ever decreasing according to  $f^p(p_{-1})$ , I need to show that the loss function (11) is still well defined. Given  $p_0 \leq p^*$ , the price level in  $t \geq 1$  periods will be  $p_t = p_0 - tZ$ . Thus, the loss function will be

$$L(p_0) = \sum_{t=1}^{\infty} \beta^{t-1} \left\{ (p_0 - tZ)^2 + \frac{\psi}{\kappa^2} (1 - \beta)^2 Z^2 \right\}$$

for  $p_0 \leq p^*$ . Expanding the price-level term inside the sum, I obtain

$$\sum_{t=1}^{\infty} \beta^{t-1} (p_0 - tZ)^2 = \frac{p_0^2}{1 - \beta} + \beta^{-1} \sum_{t=1}^{\infty} \beta^t (t^2 Z^2 - 2tZp_0).$$

The sum  $\sum_{t=1}^{\infty} \beta^{t-1} t^a$  converges for all  $a$ . Using the appropriate formulas,

$$\sum_{t=1}^{\infty} \beta^{t-1} (p_0 - tZ)^2 = \frac{p_0^2}{1 - \beta} + Z^2 \frac{1 + \beta}{(1 - \beta)^3} - \frac{2Zp_0}{(1 - \beta)^2}.$$

Thus  $L(p_0)$  is finite. For later use, the derivative of  $L$  is

$$L'(p_0) = \frac{2p_0}{1 - \beta} - \frac{2Z}{(1 - \beta)^2}.$$

Note  $L' < 0$  for all  $p_0 \leq 0$ .

Next I check that  $f^p$ ,  $f^\pi$ , and  $f^y$  satisfy the Euler equation (2) with  $R_t = -Z$  and the NKPC (1). The Euler equation is used to solve for the output gap

$$\kappa y_t = -\sigma^{-1} \kappa R_t + \kappa y_{t+1} + \sigma^{-1} \kappa \pi_{t+1}. \quad (19)$$

If  $p_0 \leq p^*$ , then  $f^p(p_{-1}) < p_{-1} \leq p^*$  and  $R_t = -Z$ . Thus

$$\kappa y_t = \sigma^{-1} \kappa Z - (1 - \beta)Z - \sigma^{-1} \kappa Z = -(1 - \beta)Z$$

as conjectured. The NKPC (1) quite trivially confirms the inflation function,

$$\pi_t = \kappa y_t + \beta \pi_{t+1} = -Z.$$

The key step is to show that  $R(p_{-1}) = -Z$  is indeed the choice of the monetary authority. If  $R_t > -Z$ , then  $p < p_{-1} - Z \leq p^*$ . The monetary authority problem can then be written as

$$\min_{p \leq p_{-1} - Z} p^2 + \frac{\psi}{\kappa^2} (p - p_{-1} + \beta Z)^2 + \beta L(p),$$

where  $L(p)$  is as given earlier. The first-order condition evaluated at  $p = p_{-1} - Z$  must satisfy

$$p_{-1} - Z - Z(1 - \beta) \frac{\psi}{\kappa^2} + \beta \left( \frac{p_{-1} - Z}{1 - \beta} - \frac{Z}{(1 - \beta)^2} \right) \leq 0.$$

This is true if

$$p_{-1} \leq (1 - \beta)Z \left( (1 - \beta) \frac{\psi}{\kappa^2} + \frac{1}{(1 - \beta)^2} \right) \equiv p^*.$$

Clearly,  $p^* > 0$ .

To complete the proof, conjecture that

$$\begin{aligned} f^p(p_{-1}) &\leq p_{-1} - Z \\ f^\pi(p_{-1}) &\leq -Z \\ f^y(p_{-1}) &\leq -(1 - \beta)Z\kappa^{-1}, \end{aligned}$$

for all  $p_{-1} > p^*$ . Note that this is only a partial characterization of the equilibrium functions. To check the conjecture, note that from (19) we have

$$\begin{aligned} \kappa y_t &= -\sigma^{-1}\kappa R_t + \kappa y_{t+1} + \sigma^{-1}\kappa\pi_{t+1} \\ &< \sigma^{-1}\kappa Z + \kappa y_{t+1} + \sigma^{-1}\kappa\pi_{t+1} \\ &\leq \sigma^{-1}\kappa Z - (1 - \beta)Z - \sigma^{-1}\kappa Z = -(1 - \beta)Z, \end{aligned}$$

where I have first used  $R_t \geq -Z$  and then the conjecture regarding  $f^\pi, f^y$ . Similarly, from the NKPC

$$\pi_t = \kappa y_t + \beta\pi_{t+1} \leq -Z.$$

It follows that the price level  $p_t$  is always below  $p_{t-1} - Z$  and, after a finite time,  $p_{t+d} < p^*$  □

### Alternative loss function for nominal GDP targeting

Proposition 5.1 also applies for a monetary authority's loss function of the form

$$l_t^p = (p_t + y_t)^2, \tag{20}$$

which, as noted in the text, is a more literal interpretation of nominal GDP targeting. The proof for Proposition 5.1 in the case of (20) follows exactly the same steps with only two minor changes in the algebra. First, the derivative of  $L$  for  $p_0 \leq p^*$  is

$$L'(p_0) = \frac{2p_0}{1 - \beta} - \frac{2Z}{(1 - \beta)^2} - 2Z\kappa^{-1}.$$

The formula for  $p^*$  is now

$$p^* = Z \left( (1 - \beta)\kappa^{-1} + 1 + \frac{\beta}{1 - \beta}(1 + \kappa^{-1}(1 - \beta))^{-1} \right).$$

### A.3 Subsection 6

#### Equilibrium characterization

The steps to characterizing an equilibrium are very similar to those in Section 4. Whenever the ZLB is binding, there is no policy decision, and thus the equilibrium condition is given by the Euler equation, or equation (7) once the output gap has been substituted for. When the ZLB is not binding, the first-order condition (14) dictates policy. After some algebra, the equilibrium condition is

$$\pi(s) \leq b_1 E_1(\pi|s) + b_2 E_j(\pi|s) + b_3 E_{j+1}(\pi|s) + b_4 \sum_{i=1}^j E_i(\pi|s) + B(s) \quad (21)$$

where

$$\begin{aligned} b_1 &= \frac{\beta\sigma^2\rho j^{-1} + \beta\psi}{\kappa^2 + \psi + \sigma^2\rho j^{-1}}, \\ b_2 &= \frac{\sigma^2\rho j^{-1}}{\kappa^2 + \psi + \sigma^2\rho j^{-1}}, \\ b_3 &= \frac{-\beta\sigma^2\rho j^{-1}}{\kappa^2 + \psi + \sigma^2\rho j^{-1}}, \\ b_4 &= \frac{\sigma\rho j^{-1}\kappa}{\kappa^2 + \psi + \sigma^2\rho j^{-1}}, \\ B(s) &= \frac{(\sigma^2\rho j^{-1} + \psi)u - \sigma^2\rho j^{-1}E_j(u|s) + \sigma\rho j^{-1}\kappa \sum_{i=1}^j E_{i-1}(v|s)}{\kappa^2 + \psi + \sigma^2\rho j^{-1}}. \end{aligned}$$

The two equations (7) and (21) are combined in (9), automatically determining whether the ZLB is binding or not.

#### Proofs

*Proof of Proposition 6.1.* Clearly  $E_1(\pi|s) = E_2(\pi|s) = E\pi$ . Equation (7) can then be simplified to

$$\pi^b(s) = (1 + \nu) E\pi + A(s).$$

Equation (21) becomes

$$\pi^u(s) = b_3 E\pi + B(s),$$

where

$$\Phi = b_1 + b_2 + b_3 + b_4 = \frac{\beta\psi + \rho(\sigma^2 j^{-1} + \sigma\kappa)}{\kappa^2 + \psi + \sigma^2\rho j^{-1}}.$$

Define  $\Gamma(x)$  as

$$\Gamma(x) = \sum_S F(s) \min \{b_3 x + B(s), (1 + \nu)x + A(s)\}.$$

$E\pi = x^*$  is an equilibrium, with

$$\pi(s) = \min \{b_3 x^* + B(s), (1 + \nu) x^* + A(s)\}$$

for all  $s \in S$ , if and only if it is a fixed point of  $\Gamma(x^*) = x^*$ .

If  $\rho > \frac{\kappa^2 + \psi(1-\beta)}{\sigma\kappa_j}$  then  $b_3 > 1$ . Since  $\nu > 0$ , the function  $h(x) = \Gamma(x) - x$  is strictly increasing. Thus, it has a single zero and there is a unique fixed point  $\Gamma(x^*) = x^*$ . Thus, there is a unique equilibrium □