When the U.S. catches a cold, Canada sneezes: a lower-bound tale told by deep learning

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Large-scale DSGE models for projection and policy analysis

Mostly used by central banks and government agencies:

- International Monetary Fund's Global Economy Model, GEM (Bayoumi et al., 2001);
- US Federal Reserve Board's SIGMA model (Erceg et al., 2006);
- Bank of Canada Terms of Trade Economic Model, ToTEM II (Dorich et al. 2013);
- European Central Bank's New Area-Wide Model, NAWM (Coenen et al. 2008);
- Bank of England COMPASS model (Burgess et al., 2013);
- Swedish Riksbank's Ramses II model (Adolfson et al., 2013).

Requirements to central bank models

1. Central bank models must mimic as close as possible the actual economies in every possible dimension.
    - *Then, the policymakers can produce realistic simulation of alternative policies and choose the best one.*
2. Central bank models must be rich and flexible enough to describe interactions between many variables of interest, including different types of foreign and domestic consumption, investment, capital, labor, prices, exchange rate, etc.
    - *Central bank models may contain hundreds of equations.*
    - *Their estimation, calibration, solution and simulation are highly nontrivial tasks.*
3. Central banks need DSGE models for policy analysis.
    - *Econometric models have limitations for policy analysis (Lucas critique).*

Numerical methods used by central banks

- Central banks use linear (first-order) perturbation methods.
  - *Advantages:*
    - ★ computationally inexpensive;
    - ★ simple to use;
    - ★ can be applied to very large problems.
  - *Drawbacks:*
    - ★ insufficiently accurate in the presence of strong nonlinearities;
    - ★ neglected second-order effects of the volatility of shocks on numerical solutions.
  - *Nonlinear effects can be economically significant:*
    - ★ Approximation errors can reach hundreds percents near ZLB; see Judd, Maliar and Maliar (Econometrica, 2017).
- In this paper, we use machine learning techniques to break the course of dimensionality and to construct accurate global nonlinear solutions.

Questions addressed in the paper

1. How large could the difference between local linear and global nonlinear solutions be in realistically calibrated central banking models?
2. Could the limitations of the first-order perturbation analysis distort policy implications of realistic central banking models?

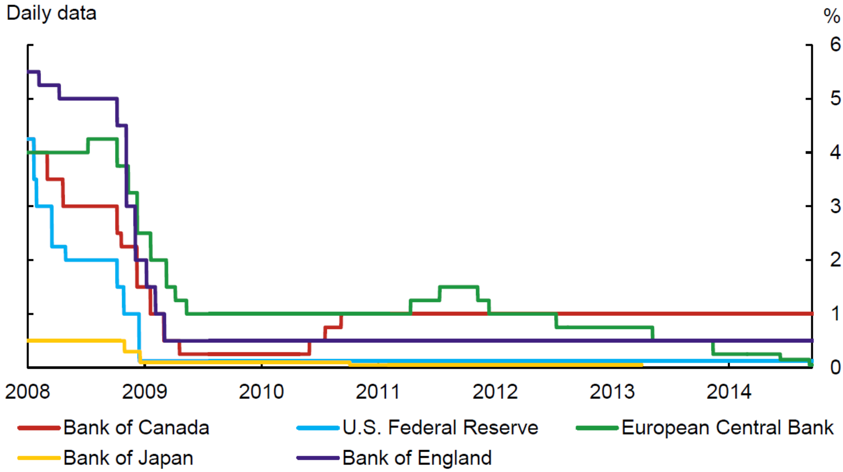
Great Recession 2007-2013 and ZLB crisis in Canada

- Canada was not at the epicentre of Great Recession.
- Unlike the U.S. and Europe, the Canadian economy did not experience a subprime crisis in 2007-2008.
- However, the contagion spread through a number of transmission channels.
- For Canada, an important transmission channel was through a direct impact on foreign trade.

"When the U.S. catches a cold, Canada sneezes".

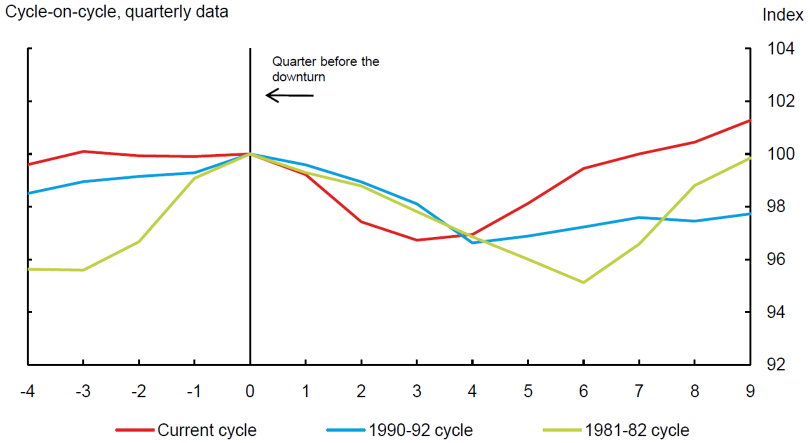
Great Recession in Canada

A sharp decline in target interest rates



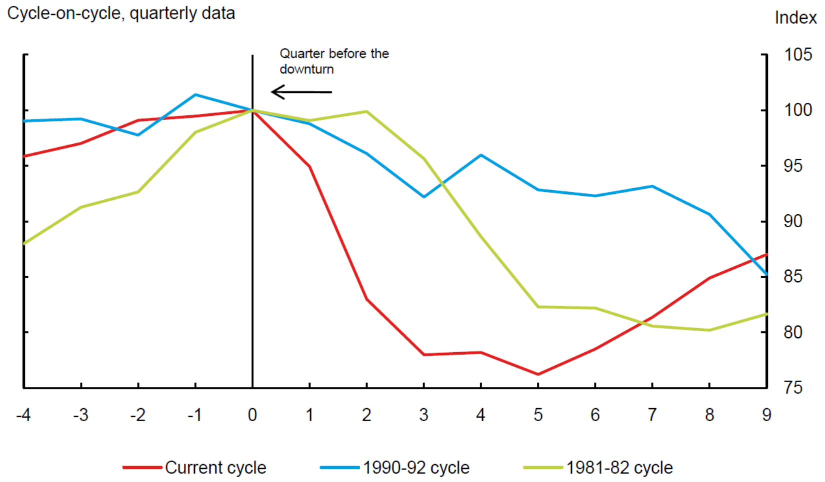
Great Recession in Canada

A sharp drop in real GDP



Great Recession in Canada

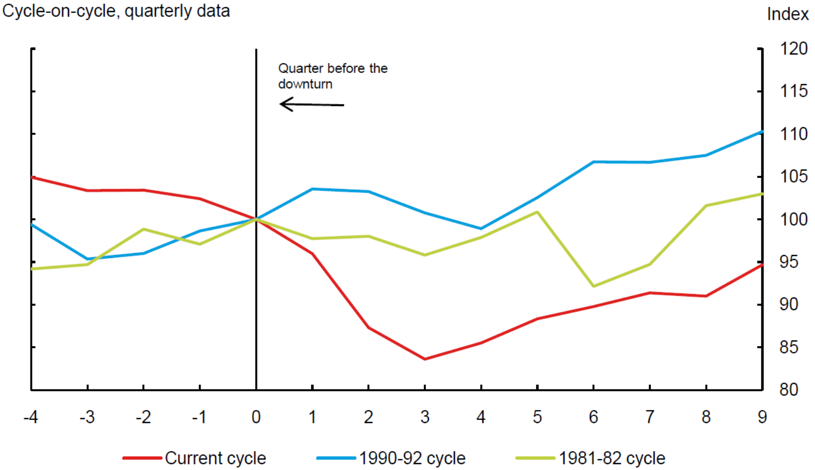
A rapid drop in real fixed investment



Sources: Statistics Canada and Bank of Canada calculations

Great Recession in Canada

A historic drop in real exports



Sources: Statistics Canada and Bank of Canada calculations

Summary of the results

1. International transmission of ELB is empirically plausible mechanism for explaining the Canadian ELB experience.
2. It is relatively easy to generate realistic ELB episodes in new Keynesian models via the foreign shocks calibrated from the data.
3. The Canadian economy would entirely avoid the ELB episode if the target inflation rate were 3 instead of 2 percent.
4. The ELB constraint plays a relatively minor role in the model performance.

Outline of the talk

1. Scaled-down version of ToTEM model
2. Numerical analysis of equilibrium:
   - *breaking the course of dimensionality with clustering analysis*
   - *capturing nonlinearities with deep learning*
3. Policy experiments:
   - *understanding the Canadian ZLB crisis*
   - *increasing of the inflation target*
4. Conclusion

Full-scale ToTEM model of Bank of Canada

- The Terms of Trade Economic Model (ToTEM) is the main projection and policy analysis model of the Bank of Canada.
- Small-open economy model
- ToTEM includes 356 equations and 215 state variables
  $\Longrightarrow$ It is too large for the existing global solution methods.

A scaled-down version of ToTEM

- We construct a scaled-down version of ToTEM, which we call a "baby ToTEM" (bToTEM) model.
- bToTEM includes 49 equations and 21 state variables
  $\Longrightarrow$ It is still a large-scale model.
- Two production sectors: final-good production and commodity production
- Meaningful trade: final goods, commodities, imports
- Households of the same type with differentiated labour services
- Taylor-type interest rate rule
- Six shocks, including exogenous rest-of-the-world (ROW) shocks

Differences between ToTEM and bToTEM

- *ToTEM*
  - 5 distinct production sectors (consumption goods and services, investment goods, government goods, noncommodity export goods, and commodities);
  - 4 sectors are identical except parameters, while the commodity sector is different;
  - A separate economic model of the rest of the world (ROW);
  - 3 types of households (they differ in their saving opportunities);
  - 8 Phillips curves.
- *bToTEM*
  - The final-good production sector is identical in structure to the consumption goods and services sector of ToTEM;
  - Linear technologies for transforming the output of this sector into other types of output that correspond to the remaining ToTEM's sectors;
  - The ROW sector is modeled using exogenous processes for foreign variables;
  - All households are of the same type;
  - 3 Phillips curves.

Production of final goods

Two stages:

1. In the first stage, intermediate goods are produced by identical perfectly competitive firms from labour, capital, commodities, and imports.
2. In the second stage, a variety of final goods are produced by monopolistically competitive firms from the intermediate goods. The variety of final goods is then aggregated into the final consumption good.

First stage of production

- Perfectly competitive firms produce an intermediate good:

$$Z_t^g = \left(\delta_l \left(A_t L_t\right)^{\frac{\sigma-1}{\sigma}} + \delta_k \left(u_t K_t\right)^{\frac{\sigma-1}{\sigma}} + \delta_{com} \left(COM_t^d\right)^{\frac{\sigma-1}{\sigma}} + \delta_m \left(M_t\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \quad (1)$$

L_t, K_t, COM_t^d = labour, capital and commodity inputs, resp., M_t = imports, u_t = capital utilization, A_t = the level of labour-augmenting technology,

$$\log\left(A_t\right) = \varphi_a \log\left(A_{t-1}\right) + \left(1-\varphi_a\right)\log\left(\bar{A}\right) + \xi_t^a. \quad (2)$$

- Capital depreciates according to the following law of motion

$$K_{t+1} = \left(1-d_t\right) K_t + I_t, \quad (3)$$

where d_t is the depreciation rate, and I_t is investment.

- The depreciation rate increases with capital utilization:

$$d_t = d_0 + \overline{d} e^{\rho(u_t - 1)}. \quad (4)$$

First stage of production (cont.)

- The firms incur a quadratic adjustment cost when adjusting the level of investment. The net output is given by

$$Z_t^n = Z_t^g - \frac{\chi_i}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2 I_t. \tag{5}$$

- The objective of the firms is to choose L_t, K_{t+1}, I_t, COM_t, M_t, u_t in order to maximize profits

$$E_0 \sum_{t=0}^{\infty} \mathcal{R}_{0,t}\left(P_t^z Z_t^n - W_t L_t - P_t^{com} COM_t^d - P_t^i I_t - P_t^m M_t\right)$$
$$\text{s.t. } (1) - (5),$$

$\mathcal{R}_{t,t+j} = \beta^j\left(\lambda_{t+j}/\lambda_t\right)\left(P_t/P_{t+j}\right) =$ stochastic discount factor.

Second stage of production

- A monopolistically competitive firm $i \in [0,1]$ produces a differentiated good

$$Z_{it} = \min\left(\frac{Z_{it}^{n}}{1-s_m}, \frac{Z_{it}^{mi}}{s_m}\right),$$

Z_{it}^{n} = intermediate good; Z_{it}^{mi} = manufactured input;
s_m = a Leontief parameter.

- The differentiated goods Z_{it} are aggregated into the final good Z_t according to a CES function:

$$Z_t = \left(\int_0^1 Z_{it}^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

- Cost minimization implies

$$Z_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} Z_t,$$

where $P_t = \left(\int_0^1 P_{it}^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}$.

Second stage of production (cont.)

- The final good is used as the manufactured inputs by each of the monopolistically competitive firms.
- Two types of monopolistically competitive firms: rule-of-thumb firms of measure ω and forward-looking firms of measure $1-\omega$.
- Within each type, with probability θ the firms index their price to the (time-varying) inflation target $\bar{\pi}_t$.
- The rule-of-thumb firms, which do not index their price in the current period, partially index their price:

$$P_{it} = (\pi_{t-1})^{\gamma} (\bar{\pi}_t)^{1-\gamma} P_{i,t-1}.$$

Second stage of production (cont.)

- The optimizing forward-looking firms solve:

$$\max_{P_t^*} E_t \left\{ \sum_{j=0}^{\infty} \theta^j \mathcal{R}_{t,t+j} \left(\prod_{k=1}^{j} \bar{\pi}_{t+k} P_t^* Z_{i,t+j} \right. \right. \\ \left. \left. - (1 - s_m) P_{t+j}^z Z_{i,t+j} - s_m P_{t+j} Z_{i,t+j} \right) \right\}$$

subject to demand constraints

$$\text{s.t.} Z_{i,t+j} = \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k} P_t^*}{P_{t+j}} \right)^{-\varepsilon} Z_{t+j}.$$

Relation between the first and second stages of production

- Production in the first and second stages are related as

$$Z_t^n = \int_0^1 Z_{it}^n di = (1 - s_m) \int_0^1 Z_{it} di = (1 - s_m) \Delta_t Z_t,$$

$\Delta_t = \int_0^1 \left(\frac{P_{it}}{P_t}\right)^{-\varepsilon} di =$ *price dispersion*.

- Law of motion of the price dispersion:

$$\Delta_t = \theta \left(\frac{\bar{\pi}_t}{\pi_t}\right)^{-\varepsilon} \Delta_{t-1} + (1-\theta)\,\omega \left(\frac{(\pi_{t-1})^{\gamma} (\bar{\pi}_t)^{1-\gamma}}{\pi_t}\right)^{-\varepsilon} \Delta_{t-1} + (1-\theta)(1-\omega)\left(\frac{P_t^*}{P_t}\right)^{-\varepsilon}.$$

- Investment goods and noncommodity exports are produced from the final goods according to linear technology, $P_t^i = \iota_i P_t$ and $P_t^{nc} = \iota_x P_t$.

Imports

- Intermediate imported goods M_{it} are bounded into the final imported good M_t according to
$$M_t = \left(\int_0^1 M_{it}^{\frac{\varepsilon_m - 1}{\varepsilon_m}} di \right)^{\frac{\varepsilon_m}{\varepsilon_m - 1}}.$$
- The demand for an intermediate imported good i:
$$M_{it} = \left(\frac{P_{it}^m}{P_t^m} \right)^{-\varepsilon_m} M_t,$$
where $P_t^m = \left(\int_0^1 \left(P_{it}^m \right)^{1-\varepsilon_m} di \right)^{\frac{1}{1-\varepsilon_m}}$.
- The price of intermediate imported goods is set in the currency of the importing country.

Import (cont.)

- The optimizing forward-looking firms solves

$$\max_{P_t^{m*}} E_t \left[\sum_{j=0}^{\infty} (\theta_m)^j \mathcal{R}_{t,t+j} \left(\prod_{k=1}^{j} \bar{\pi}_{t+k} P_t^{m*} M_{i,t+j} - e_{t+j} P_{t+j}^{mf} M_{i,t+j} \right) \right]$$

subject to demand constraints

$$M_{i,t+j} = \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k} P_t^{m*}}{P_{t+j}^{m}} \right)^{-\varepsilon_m} M_{t+j},$$

P_t^{mf} = foreign price of imports; e_t = nominal exchange rate (domestic price of a unit of foreign currency).

Foreign demand and foreign economy

- The foreign demand for Canadian noncommodity exports is given by the demand function

$$X_t^{nc} = \gamma^f \left(\frac{P_t^{nc}}{e_t P_t^f} \right)^{-\varphi} Z_t^f,$$

$P_t^{nc}/e_t =$ foreign price of noncommodity exports;
$P_t^f =$ foreign general price level.

- The balance of payments

$$\frac{e_t B_t^f}{R_t^f \left(1 + \kappa_t^f\right)} - e_t B_{t-1}^f = P_t^{nc} X_t^{nc} + P_t^{com} X_t^{com} - P_t^m M_t,$$

$B_t^f =$ domestic holdings of foreign-currency denominated bonds;
$\kappa_t^f =$ risk premium.

Foreign demand and foreign economy (cont.)

- The rest of the world is specified by three exogenous processes that describe the evolution of foreign variables.
- The foreign demand for Canadian noncommodity exports Z_t^f

$$\log\left(Z_t^f\right) = \varphi_{Zf} \log\left(Z_{t-1}^f\right) + \left(1-\varphi_{zf}\right)\log\left(\bar{Z}^f\right) + \xi_t^{zf}.$$

- A foreign interest rate shock r_t^f

$$\log\left(r_t^f\right) = \varphi_{rf} \log\left(r_{t-1}^f\right) + \left(1-\varphi_{rf}\right)\log\left(\bar{r}\right) + \xi_t^{rf}.$$

- A foreign commodity price p_t^{comf} is

$$\log\left(p_t^{comf}\right) = \varphi_{comf} \log\left(p_{t-1}^{comf}\right) + \left(1-\varphi_{comf}\right)\log\left(\bar{p}^{comf}\right) + \xi_t^{comf},$$

$\xi_t^{zf}, \xi_t^{rf}, \xi_t^{comf}$ = normally distributed random variables;
$\varphi_{Zf}, \varphi_{rf}, \varphi_{comf}$ = autocorrelation coefficients.

Commodity production

- The commodities are produced from the final goods by representative, perfectly competitive domestic firms:

$$COM_t = \left(Z_t^{com}\right)^{s_z}\left(A_t F\right)^{1-s_z} - \frac{\chi_{com}}{2}\left(\frac{Z_t^{com}}{Z_{t-1}^{com}} - 1\right)^2 Z_t^{com},$$

  $F =$ a fixed production factor (land).
  Here, the second term is quadratic adjustment costs.

- The commodities are sold at the rest-of-the-world price adjusted by the nominal exchange rate

$$P_t^{com} = e_t P_t^{com\,f}.$$

- The commodities are sold domestically (COM_t^d) or exported to the rest of the world (X_t^{com})

$$COM_t = COM_t^d + X_t^{com}.$$

Households

- The representative household's period utility function:

$$U_t = \frac{\mu}{\mu - 1} \left(C_t - \xi \bar{C}_{t-1}\right)^{\frac{\mu-1}{\mu}} \exp\left(\frac{\eta \left(1-\mu\right)}{\mu \left(1+\eta\right)} \int_0^1 \left(L_{ht}\right)^{\frac{\eta+1}{\eta}} dh\right) \eta_t^c,$$

  C_t = consumption of finished goods;
  $\bar{C}_t$ = aggregate consumption;
  L_{ht} = labour service of type h;
  η_t^c = a consumption demand shock.

- We assume

$$\log\left(\eta_t^c\right) = \varphi_c \log\left(\eta_{t-1}^c\right) + \xi_t^c,$$

  ξ_t^c = a normally distributed variable; φ_c = an autocorrelation coefficient.

Households (cont.)

- The representative household solves

$$E_t\left[\sum_{j=0}^{\infty}\beta^t U_{t+j}\right]$$

$$\text{s.t. } P_tC_t+\frac{B_t}{R_t}+\frac{e_tB_t^f}{R_t^f\left(1+\kappa_t^f\right)}=B_{t-1}+e_tB_{t-1}^f+\int_0^1 W_{ht}L_{ht}dh+\Pi_t,$$

B_t = holdings of domestic bonds;
B_t^f = holdings of foreign-currency denominated bonds;
Π_t are profits paid by the firms.

- To induce the stationarity of the model, we assume that the risk premium κ_t^f is

$$\kappa_t^f=\varsigma\left(\bar{b}^f-b_t^f\right),$$

$b_t^f=e_tB_t^f/\left(\pi_{t+1}^f P_t\bar{Y}\right)$ = normalized bond holdings; see Schmitt-Grohé and Uribe (2003).

Labour unions

- The representative household supplies a variety of differentiated labour service L_{ht}, $h \in [0,1]$ to the labour market.
- The differentiated labour service is aggregated according to

$$L_t = \left(\int_0^1 L_{ht}^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh \right)^{\frac{\varepsilon_w}{\varepsilon_w - 1}} .$$

- L_t is used in the first stage of production.
- Cost minimization implies the demand

$$L_{ht} = \left(\frac{W_{ht}}{W_t} \right)^{-\varepsilon_w} L_t,$$

$W_{ht} =$ wage for labour of type h; $W_t \equiv \left(\int_0^1 W_{ht}^{1-\varepsilon_w} dh \right)^{\frac{1}{1-\varepsilon_w}}$.

Labour unions (cont.)

- Two types: rule-of-thumb (measure ω_w) and forward-looking (measure $1-\omega_w$) unions.
- Within each type, with probability θ_w a union indexes wage to the inflation target $\bar{\pi}_t$, $W_{it} = \bar{\pi} W_{i,t-1}$.
- The rule-of-thumb unions which do not index their wage set

$$W_{it} = \left(\pi_{t-1}^{w}\right)^{\gamma_w} \left(\bar{\pi}_t\right)^{1-\gamma_w} W_{i,t-1}.$$

- The forward-looking unions that do not index their wage solve:

$$\max_{W_t^*} E_t \left[\sum_{j=0}^{\infty} \left(\beta\theta_w\right)^j U_{t+j} \right]$$

$$\text{s.t. } L_{h,t+j} = \left(\frac{\prod_{k=1}^{j} \bar{\pi}_{t+k} W_t^*}{W_{t+j}} \right)^{-\varepsilon_w} L_{t+j},$$

$$P_{t+j} C_{t+j} = \prod_{k=1}^{j} \bar{\pi}_{t+k} W_t^* L_{h,t+j} dh + ...$$

Monetary policy

- Taylor rule:

$$R_t^n = \rho_r R_{t-1} + (1 - \rho_r) \left[\bar{R} + \rho_\pi (\pi_t - \bar{\pi}_t) + \rho_Y \left(\log Y_t - \log \bar{Y}_t \right) \right] + \eta_t^r$$

$\bar{Y}_t$ = potential output; η_t^r = interest rate shock,

$$\eta_t^r = \varphi_r \eta_{t-1}^r + \xi_t^r,$$

ξ_t^r = a normally distributed variable; φ_r = autocorrelation coefficient.

- Potential output changes with productivity as

$$\log \bar{Y}_t = \varphi_{zf} \log \bar{Y}_{t-1} + (1 - \varphi_{zf}) \log \left(\frac{A_t \bar{Y}}{\bar{A}} \right).$$

- If an effective lower bound (ELB) is imposed on the nominal interest rate, R_t^{ELB}, then

$$R_t = \max \left\{ R_t^n, R_t^{ELB} \right\}.$$

Market clearing conditions

- Resource feasibility condition

$$Z_t = C_t + \iota_i I_t + \iota_x X_t^{nc} + Z_t^{com} + \upsilon_z Z_t.$$

- GDP

$$Y_t = C_t + I_t + X_t^{nc} + X_t^{com} - M_t + \upsilon_y Y_t.$$

- GDP deflator

$$P_t^y Y_t = P_t C_t + P_t^i I_t + P_t^{nc} X_t^{nc} + P_t^{com} X_t^{com} - P_t^m M_t + \upsilon_y p_t^y Y_t.$$

A list of endogenous model variables

#	Variable	Notation	In logarithms?
1	labour input	L_t	yes
2	capital input	K_t	yes
3	investment	I_t	yes
4	commodities used domestically	COM_t^d	yes
5	import	M_t	yes
6	capital utilization	u_t	no
7	capital depreciation	d_t	no
8	gross production of intermediate good	Z_t^g	yes
9	net production of intermediate good	Z_t^n	yes
10	total production	Z_t	yes
11	consumption	C_t	yes
12	marginal utility of consumption	λ_t	yes
13	nominal interest rate	R_t	no
14	inflation	π_t	yes
15	consumption Phillips curve term	F_{1t}	yes
16	consumption Phillips curve term	F_{2t}	yes
17	price dispersion	Δ_t	yes
18	real marginal cost	rmc_t	yes

A list of endogenous model variables (cont.)

#	Variable	Notation	In logarithms?
19	inflation target	$\bar{\pi}_t$	yes
20	real price of intermediate good	p_t^z	yes
21	real price of import	p_t^m	yes
22	foreign price of import	p_t^{mf}	yes
23	real exchange rate	s_t	yes
24	imported good inflation	π_t^m	yes
25	imports Phillips curve term	F_{1t}^m	yes
26	imports Phillips curve term	F_{2t}^m	yes
27	wage inflation	π_t^w	yes
28	wage Phillips curve term	F_{1t}^w	yes
29	wage Phillips curve term	F_{2t}^w	yes
30	wage dispersion	Δ_t^w	yes
31	real wage	w_t	yes
32	optimal wage	w_t^*	yes
33	real price of commodities	p_t^{com}	yes
34	marginal product of capital	MPK_t	yes
35	interest rate on capital	R_t^k	no
36	real price of investment	p_t^i	yes

A list of endogenous model variables (cont.)

#	Variable	Notation	In logarithms?
37	Tobin's Q	q_t	yes
38	price of non-commodity export	p_t^{xz}	yes
39	interest premium on foreign bonds	κ_t^f	no
40	non-commodity export	X_t^{nc}	yes
41	export of commodities	X_t^{com}	yes
42	total commodities produced	COM_t	yes
43	final goods used in commodity production	Z_t^{com}	yes
44	GDP	Y_t	yes
45	potential GDP	$\bar{Y}_t$	yes
46	GDP deflator	p_t^y	yes
47	holdings of foreign bonds in real terms	b_t^f	no
48	auxiliary expectation term	ex_t^i	no
49	auxiliary expectation term	ex_t^{com}	no

Calibration

- 61 parameters
- Whenever possible, we use the same parameters as in ToTEM.
- We choose the remaining parameters to reproduce observations on the Canadian economy.
- We target the ratios of the following variables to nominal GDP:
    - consumption,
    - investment,
    - noncommodity export,
    - commodity export,
    - import,
    - total commodities,
    - labor input.

Parameters in endogenous model equations

Parameter	Symbol	Value	Source
Rates			
– real interest rate	$\bar{r}$	1.0076	ToTEM
– discount factor	β	0.9925	ToTEM
– inflation target	$\bar{\pi}$	1.005	ToTEM
– nominal interest rate	$\bar{R}$	1.0126	ToTEM
– ELB on the nominal interest rate	R^{ELB}	1.01	fixed
Output production			
– CES elasticity of subtitution	σ	0.5	ToTEM
– CES labor share paramater	δ_L	0.249	calibrated
– CES capital share parameter	δ_K	0.575	calibrated
– CES commodity share parameter	δ_{COM}	0.0015	calibrated
– CES import share parameter	δ_M	0.0287	calibrated
– investment adjustment cost	χ_I	20	calibrated
– fixed depreciation rate	d_0	0.0054	ToTEM
– variable depreciation rate	$\bar{d}$	0.0261	ToTEM
– depreciation semielasticity	ρ	4.0931	calibrated
– real investment price	ι_I	1.2698	ToTEM
– real noncommodity export price	ι_X	1.143	ToTEM
– labour productivity	$\bar{A}$	100	normalization

Parameters in endogenous model equations (cont.)

Parameter	Symbol	Value	Source
Price setting parameters for consumption			
– probability of indexation	θ	0.75	ToTEM
– RT indexation to past inflation	γ	0.0576	ToTEM
– RT share	ω	0.4819	ToTEM
– elasticity of substitution of consumption goods	ε	11	ToTEM
– Leontief technology parameter	s_m	0.6	ToTEM
Price setting parameters for imports			
– probability of indexation	θ^m	0.8635	ToTEM
– RT indexation to past inflation	γ^m	0.7358	ToTEM
– RT share	ω^m	0.3	ToTEM
– elasticity of substitution of imports	ε^m	4.4	ToTEM
Price setting parameters for wages			
– probability of indexation	θ^w	0.5901	ToTEM
– RT indexation to past inflation	γ^w	0.1087	ToTEM
– RT share	ω^w	0.6896	ToTEM
– elasticity of substitution of labour service	ε^w	1.5	ToTEM

Parameters in endogenous model equations (cont.)

Parameter	Symbol	Value	Source
Household utility			
– consumption habit	ξ	0.9396	ToTEM
– consumption elasticity of substitution	μ	0.8775	ToTEM
– wage elasticity of labor supply	η	0.0704	ToTEM
Taylor rule			
– interest rate persistence parameter	ρ_r	0.83	ToTEM
– interest rate response to inflation gap	ρ_π	4.12	ToTEM
– interest rate response to output gap	ρ_y	0.4	ToTEM
Other			
– capital premium	κ^k	0.0674	calibrated
– exchange rate persistence parameter	$\varkappa$	0.1585	ToTEM
– foreign commodity price	$\bar{p}^{COMf}$	1.6591	ToTEM
– foreign import price	$\bar{p}^{Mf}$	1.294	ToTEM
– risk premium response to debt	ς	0.0083	calibrated
– foreign demand elasticity	ϕ	0.4	calibrated
– elasticity in commodity production	s_z	0.8	calibrated
– land	F	0.1559	calibrated
– share of other components of output	υ_Z	0.7651	calibrated
– share of other components of GDP	υ_Y	0.311	calibrated

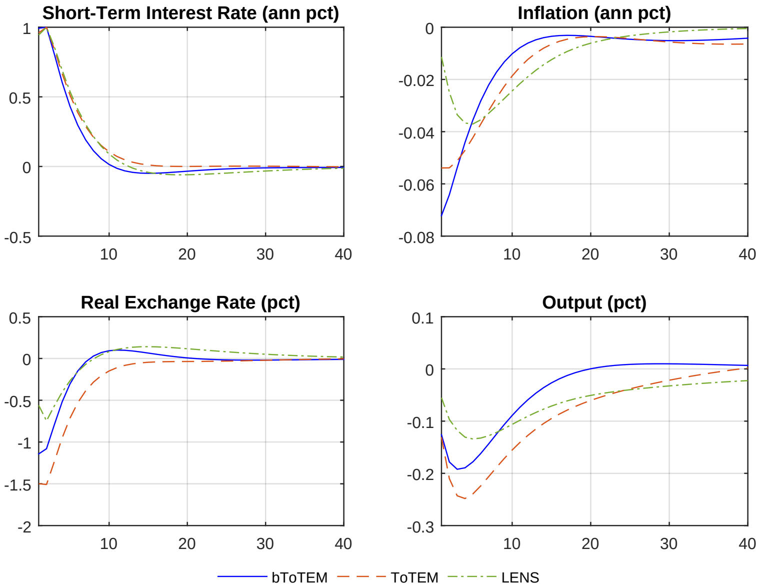
Parameters in exogenous model equations

Parameter	Symbol	Value	Source
Shock persistence			
– persistence of interest rate shock	φ_R	0.25	ToTEM
– persistence of productivity shock	φ_A	0.9	fixed
– persistence of consumption demand shock	φ_C	0	fixed
– persistence of foreign output shock	φ_{Zf}	0.9	fixed
– persistence of foreign commodity price shock	φ_{pCOMf}	0.87	calibrated
– persistence of foreign interest rate shock	φ_{rf}	0.88	calibrated
Shock volatilities			
– volatility of interest rate shock	σ_R	0.0006	calibrated
– volatility of productivity shock	σ_A	0.0067	calibrated
– volatility of consumption demand shock	σ_C	0.0001	fixed
– volatility of foreign output shock	σ_{Zf}	0.0085	calibrated
– volatility of foreign commodity price shock	σ_{pCOMf}	0.0796	calibrated
– volatility of foreign interest rate shock	σ_{rf}	0.002	calibrated

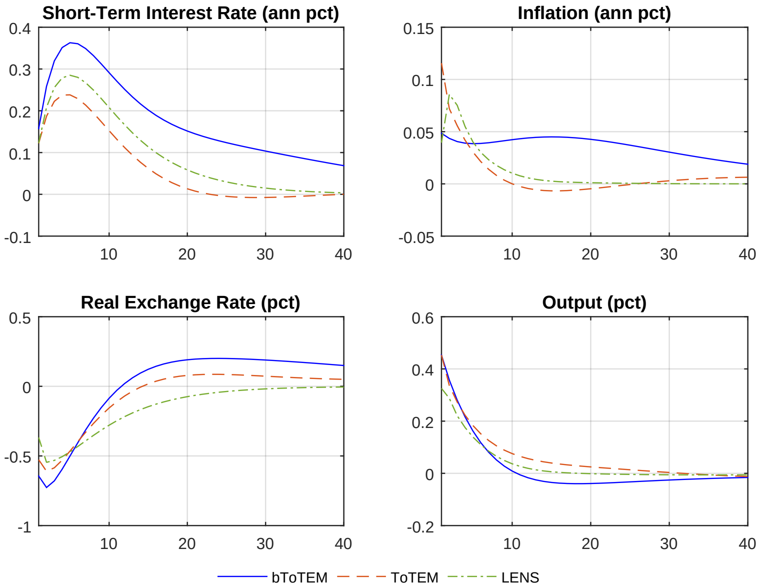
A comparison of bToTEM to ToTEM and LENS

- The Bank of Canada uses a first-order perturbation method to solve ToTEM.
- For ToTEM, we use IRIS – open-source software used by the Bank of Canada for macroeconomic modeling.
- For bToTEM, we use IRIS Toolbox, as well as Dynare.
- We checked that the IRIS and Dynare packages produce indistinguishable numerical solutions for bToTEM.
- Also, we include for comparison the LENS model – another model of the Bank of Canada.
- LENS is a semistructural model.
- Both, ToTEM and LENS models, include more shocks than bToTEM:
  - 52 shocks in ToTEM and 98 shocks in LENS.

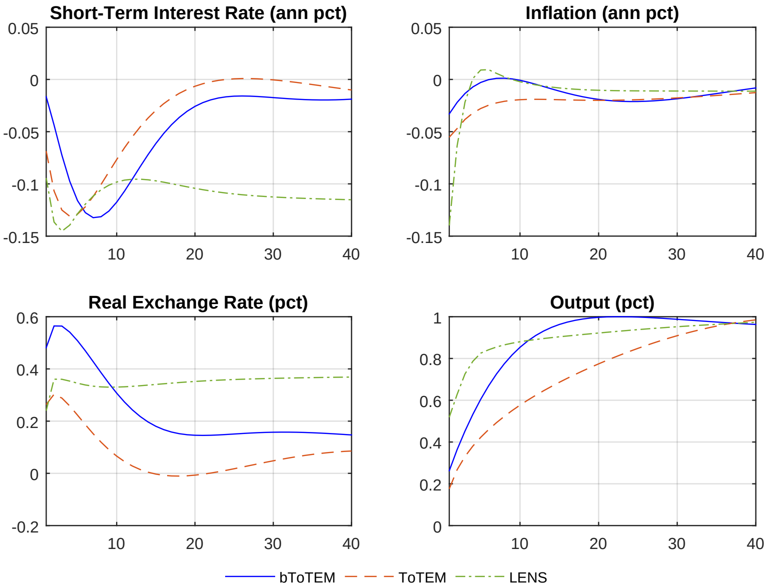
Impulse response to an interest rate shock



Impulse response to a consumption demand shock



Impulse response to a permanent productivity shock



bTоTEM: a serious challenge for global methods

- The models like bToTEM has not been yet studied in the literature using global methods.
- bToTEM contains 21 state variables (6 exogenous and 15 endogenous) $\Longrightarrow$ *curse of dimensionality*.
- Moreover, the bToTEM's equations are complex and require the use of numerical solvers.
- To solve the bToTEM model, we use an ergodic set method.

Advantages of ergodic set methods

- Conventional projection methods operate on exogenously given hypercube.
- However, many areas of the hypercube have low probability of occurrence - we might not need to know the solution in low probability areas.
- Stochastic simulation methods construct the solution on a set of simulated points where the solution "lives".
- This can save on cost a lot!

Illustrative example: a representative-agent model

The representative-agent neoclassical stochastic growth model

$$\max_{\{k_{t+1},c_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

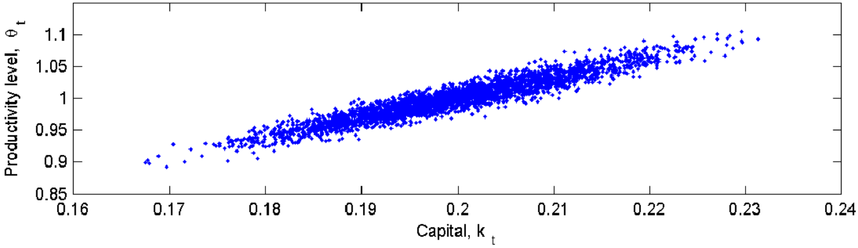
$$\text{s.t.} \quad c_t + k_{t+1} = (1-\delta) k_t + \theta_t f(k_t),$$

$$\ln \theta_{t+1} = \rho \ln \theta_t + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim \mathcal{N}\left(0, \sigma^2\right),$$

where initial condition (k_0, θ_0) is given;
$u(\cdot)$ = utility function; $f(\cdot)$ = production function;
c_t = consumption; k_{t+1} = capital; θ_t = productivity;
β = discount factor; δ = depreciation rate of capital;
ρ = autocorrelation coefficient of the productivity level;
σ = standard deviation of the productivity shock.

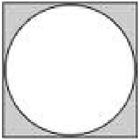
Advantages of ergodic set methods

A stochastic simulation methods: "Grid" is adaptive: we solve the model only in the area of the state space that is visited in simulation.



Reduction in cost in a 2-dimensional case

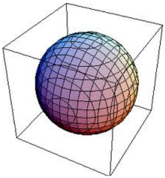
- How much can we save on cost using the ergodic-set domain comparatively to the hypercube domain?
- Suppose the ergodic set is a circle (it was an ellipse in the figure).
- In the 2-dimensional case, a circle inscribed within a square occupies about 79% of the area of the square.
- The reduction in cost is proportional to the shaded area in the figure.



- It does not seem to be a large gain.

Reduction in cost in a d-dimensional case

- In a 3-dimensional case, the gain is larger (a volume of a sphere of diameter 1 is 52% of the volume of a cube of width 1)



- In a d-dimensional case, the ratio of a hypersphere's volume to a hypercube's volume

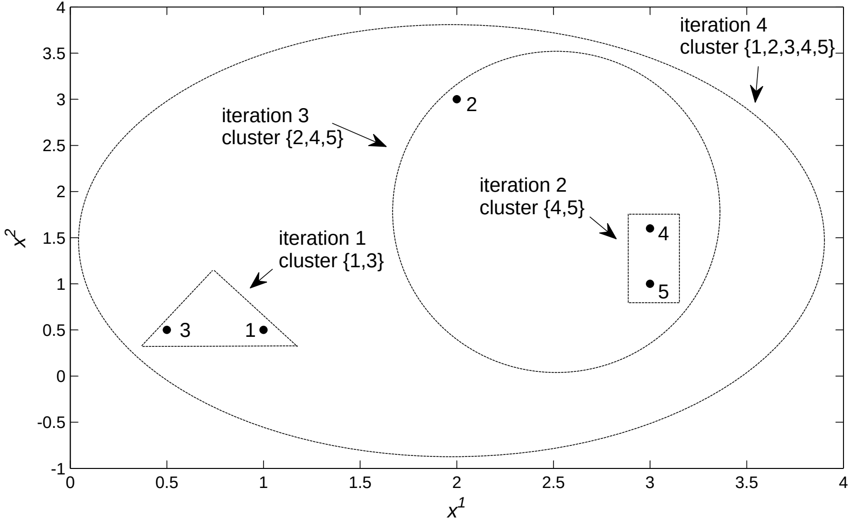
$$\mathcal{V}^d = \begin{cases} \frac{(\pi/2)^{\frac{d-1}{2}}}{1\cdot 3\cdot \ldots \cdot d} & \text{for } d = 1, 3, 5... \\ \frac{(\pi/2)^{\frac{d}{2}}}{2\cdot 4\cdot \ldots \cdot d} & \text{for } d = 2, 4, 6... \end{cases}.$$

- $\mathcal{V}^d$ declines very rapidly with dimensionality of state space. When $d = 10 \Rightarrow \mathcal{V}^d = 3 \cdot 10^{-3}$ (0.3%). When $d = 30 \Rightarrow \mathcal{V}^d = 2 \cdot 10^{-14}$.
- We face a tiny fraction of cost we would have faced on the hypercube.

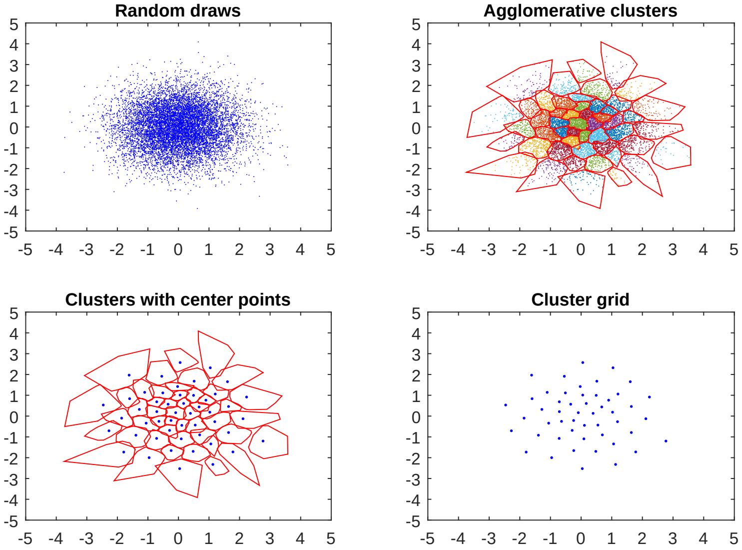
Unsupervised learning: clustering analysis

- Cluster grid algorithm is an ergodic set algorithm of Maliar and Maliar (2015).
- It is a projection-style global solution method that uses adaptive grid:
  - the model is solved only in the area of the state space visited in simulation
  - relies on integration and optimization methods are tractable in high-dimensional problems
- CGA can accurately solve models with dozens of state variables.

Example: agglomerative hierarchical clustering algorithm



Example: Construction of a cluster grid

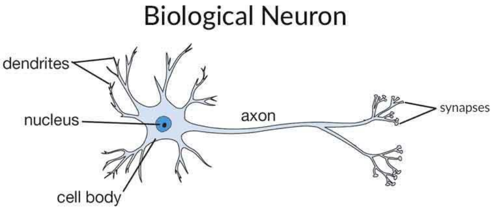


Supervised learning: deep neural network

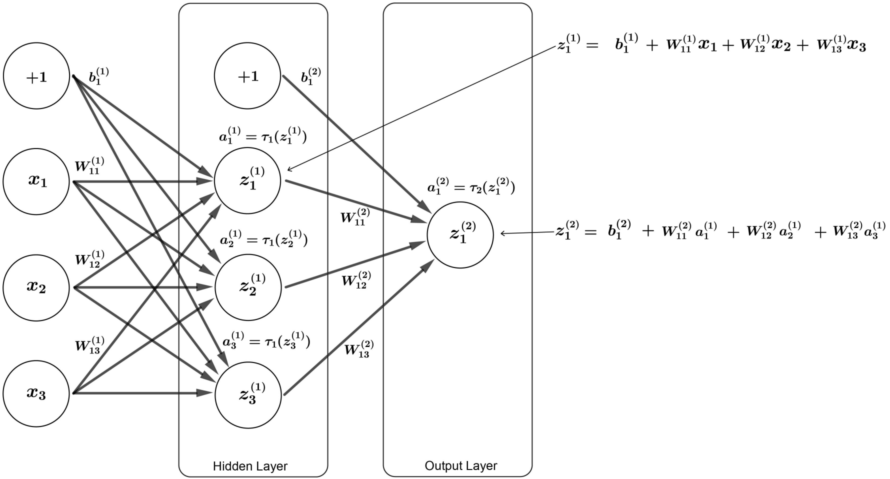
- Initially, we constructed nonlinear solutions using second-order polynomial approximation.
- But second-degree polynomial is not flexible enough to accurately approximate highly non-linear models like bToTEM.
- The difference between perturbation and global solutions cannot be too large as both of them are built using the same quadratic approximation function.
- To approximate nonlinearities more accurately, one needs more flexible approximating functions.
- We now use neural networks and deep learning techniques to produce more flexible approximations.

Artificial neural networks

- Origins: Algorithms that try to mimic the brain.
- Was very widely used in 80s and early 90s; popularity diminished in late 90s.
- Recent resurgence: State-of-the-art technique for many applications.



Example: a neural network



Neural network in bToTEM

- Three layers: input layer, hidden layer, output layer
- Quadratic basis (terms of the second-order ordinary polynomial) of the state variables as an input.
- In the hidden layer, as an activation function, we use a symmetric sigmoid function, namely, the hyperbolic tangent function
- The neutral network includes 2,915 coefficients: 2,783 coefficients in the hidden layer and 132 coefficients in the output layer.

Understanding the role of nonlinearities in the solution

Potential effects of nonlinearities on the properties of the solution compared to a plain linearization method:

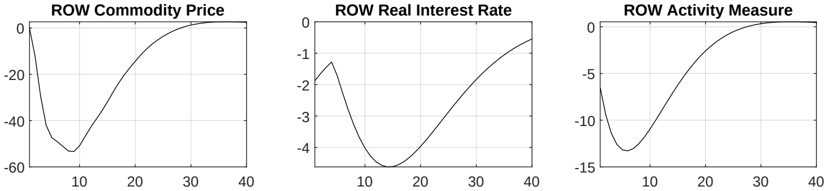
- **(Uncertainty effect).** Linearized solutions do not depend on the degree of volatility, as do nonlinear solutions.
- **(High-order effect).** Linearization method neglects high-order polynomial terms, unlike more flexible nonlinear solutions.
- **(Solution-domain effect).** Perturbation (local) solutions are constructed to be accurate in a deterministic steady state, and their accuracy can deteriorate dramatically when deviating from the steady state, in particular, in the area of ELB.

Experiment 1: Foreign-driven recession

- The U.S. is the main Canadian trade partner (around 75% of Canadian exports goes to the US).
- In 2008, the Canadian economy experienced a huge 16% drop in exports.
- In 2009–2010, the Bank of Canada targeted the overnight interest rate at 0.25% annually (the lower bound).
- To model the ROW:
  - we produce impulse responses for 3 ROW variables (commodity price, interest rate, foreign activity measure) from ToTEM.
  - we use them as exogenous shocks in the bToTEM model.
- ROW activity measure in ToTEM declines by 7% on the impact of shock and by 12% at the peak – the numbers consistent with the magnitudes during the Great Recession.

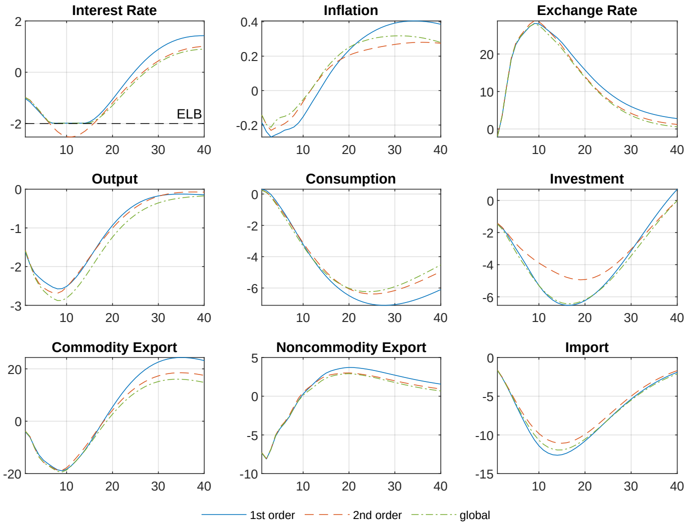
Experiment 1: Foreign-driven recession

Exogenous ROW shocks



Experiment 1: Foreign-driven recession

Linear local, quadratic local and global solutions



Approximation errors

Residuals in the model's equations on the simulated path, $\log 10$ units

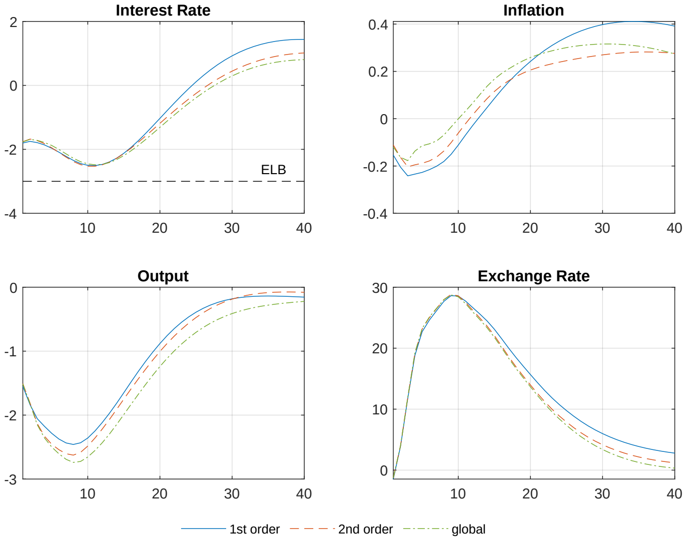
	Maximum residual			Average residual		
	Local 1st order	Local 2nd order	Global DL	Local 1st order	Local 2nd order	Global DL
R_t	-3.82	-3.91	-4.27	-4.51	-4.90	-4.74
π_t^m	-2.48	-2.70	-3.99	-3.60	-3.76	-5.00
s_t	-2.40	-1.98	-3.05	-3.39	-3.15	-3.97
Y_t	-2.58	-3.17	-3.96	-3.24	-3.96	-4.96
C_t	-3.19	-3.11	-4.01	-3.95	-4.23	-5.11
I_t	-3.01	-3.38	-3.52	-4.42	-4.50	-4.84
X_t^{nc}	-2.80	-2.38	-3.45	-3.79	-3.55	-4.37
X_t^{com}	-1.76	-2.29	-3.18	-2.51	-3.07	-4.41
M_t	-2.16	-2.94	-3.60	-2.89	-4.05	-4.65
Average	-2.75	-2.76	-3.62	-3.58	-3.86	-4.63
Max	-1.44	-1.48	-2.37	-2.51	-2.67	-3.60

Experiment 2: Higher inflation target

- For the last 25 years, the Bank of Canada adhered to the inflation targeting, however, every three to five years it revises their inflation-control framework.
- Current inflation target is 2%.
- A higher inflation target could be beneficial by reducing frequency and severity of ELB episodes.
- Kryvtsov and Mendes (2015), Dorich et al. (2017).
- We use the bToTEM model to assess the effects of an increase in the inflation target from 2% to 3%.

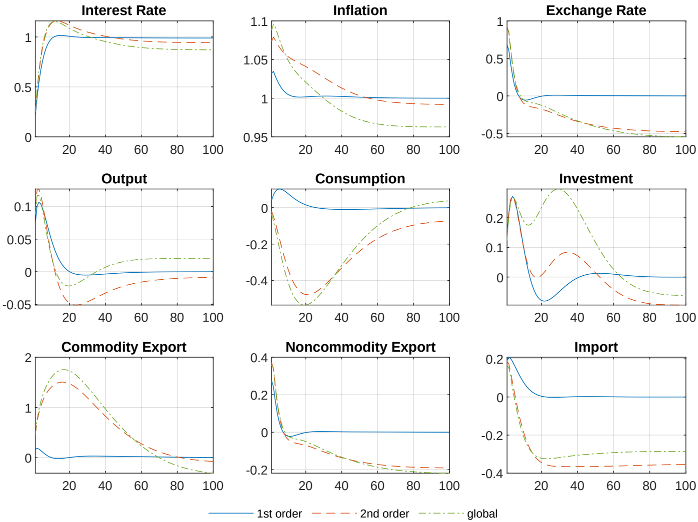
Experiment 2: Higher inflation target

A 3% inflation target could prevent ELB episodes similar to the 2009-10 episode



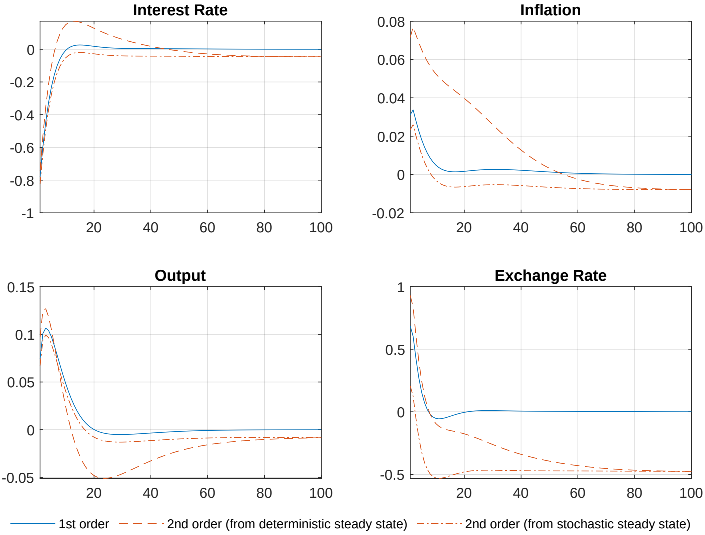
Experiment 2: Higher inflation target

Transition from the deterministic steady state with the inflation target of 2%



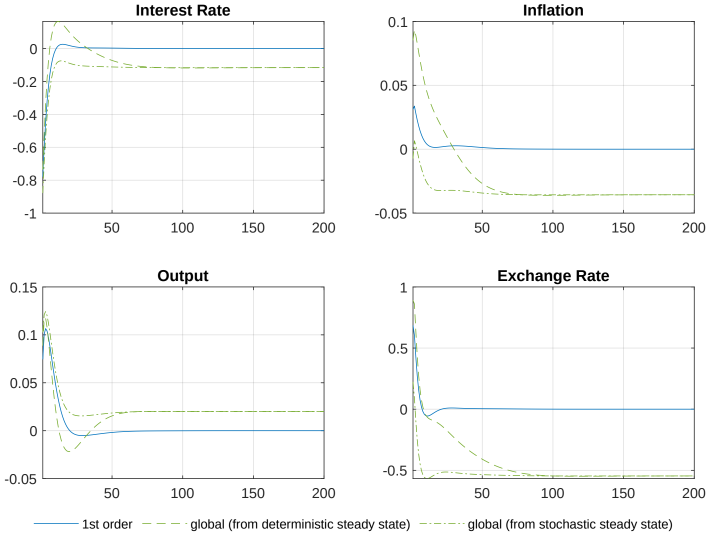
Experiment 2: Higher inflation target

Transition from the different states with the target of 2%, 2nd order solution



Experiment 2: Higher inflation target

Transition from the different states with the target of 2%, global solution



Closing condition

Revisiting the stationarity condition of Schmitt-Grohe and Uribe (2003):

- closing condition in an exponential form, used in Schmitt-Grohe and Uribe (2003)

$$\kappa_t^f = \varsigma \left[\exp\left(\bar{b}^f - b_t^f \right) - 1 \right]$$

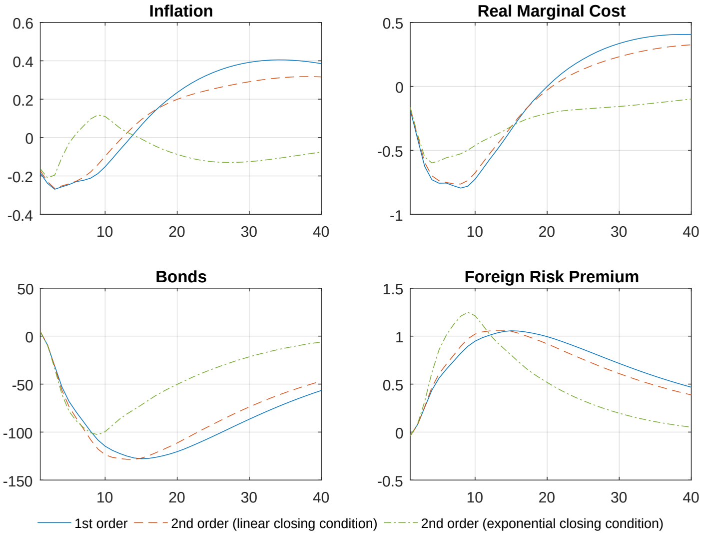
- linear closing condition

$$\kappa_t^f = \varsigma \left(\bar{b}^f - b_t^f \right)$$

- the two conditions are equivalent up to the first order

Nonlinearities are important, finally!

Transition under different closing conditions



Conclusion

- bTоTEM model
  - could be used by Bank of Canada as an alternative model for projection and policy analysis, complement the main ToTEM model
  - tractable with global nonlinear solution methods
  - accuracy can be assessed
- Machine learning techniques
  - open a new era in quantitative macroeconomics and policy analysis
  - allow to solve both accurately and reliably large-scale models intractable up to now
- Linear vs. nonlinear solutions
  - the role of nonlinearities is modest in the benchmark bToTEM model
  - but apparently innocent changes in the assumptions like a closing condition can make nonlinearity important
- Modeling of ZLB
  - surprisingly easy to generate ZLB episodes in an open-economy setting
  - the recent ZLB episode in Canada can be modelled through ROW shocks