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Optimal Time-Consistent Government Debt Maturity, Fiscal Policy, and Default[1]

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Abstract

I develop a tractable model to study the optimal debt maturity structure and fiscal policy in an environment with incomplete markets, lack of commitment, and opportunity to default by the government. The default on public debt is endogenous and the real interest rate reflects the default risk and the marginal rate of substitution between present and future consumption. I show that the Lucas and Stokey (1983) time-consistency result can be extended to environments with an opportunity of outright default. The maturity is used to resolve the time-consistency problem: The present government can incentivize future governments to stick to an ex ante optimal sequence of fiscal policies and interest rates. I show that if both risk-free interest rates and risk premiums can be manipulated, the optimal maturity structure tends to have a decaying profile: The government issues debt at all maturity dates, but the distribution of payments over time is skewed toward the short end. The model allows for numerical characterization of the optimal maturity structure of debt with arbitrarily large number of maturities. Debt maturity data across countries are consistent with model predictions.

Keywords: Time consistency, maturity structure, sovereign debt, fiscal policy, default.

JEL classification: E43, E62, F22, F34

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1. Introduction

Time consistency of optimal fiscal policy has been extensively discussed in the literature. As shown by Lucas and Stokey (1983), the time-consistency problem arises as a government lacks commitment to fiscal policies and can manipulate the value of outstanding debt by altering risk-free interest rates. However, in a real economy with no outright default, the problem can be resolved by carefully choosing a unique maturity structure of government state-contingent debt. Moreover, Debortoli, Nunes, and Yared (2017) show that even if state-contingent bonds are absent, debt maturity is used to minimize the costs of lack of commitment (not the costs of lack of insurance) and the optimal maturity is approximately flat. A similar time-consistency problem arises in open economy models with default being an option. Sovereigns can alter the value of outstanding debt by manipulating default risks. As shown by Aguiar et al. (2019), in a model with endogenous default but exogenous risk-free interest rates, the time-consistency problem can be resolved by selecting the right maturity structure. However, the conclusion regarding the optimal shape of maturity is quite the opposite: the government issues only short-term debt and abstains from any active issuance or repurchase of long-term liabilities.

In this paper, I combine both sources of time inconsistency – manipulation of risk-free interest rates and risks of default - within a unified framework. I develop a tractable model to study the optimal fiscal policy and optimal debt maturity structure in an environment with incomplete markets, lack of commitment to fiscal policies, and endogenous default on public debt. The finding is that if a government can alter both risk-free rates and default premiums, time consistency of fiscal policy can be achieved and optimal maturity structure exhibits a decaying profile: total payments due at a later maturity date are lower. This prediction is in line with empirical data: the observed term structures of most countries are neither flat nor short but skewed toward the short end.

The model features a benevolent government and a continuum of atomistic households with strictly concave utility functions over private consumption. The government conducts fiscal policy choosing budget deficit or surplus as well as the maturity structure of its debt. Households are the only lenders to the government. The government cannot commit to either future fiscal policies or payment of its debt. The markets are incomplete, and the set of financial instruments is limited to bonds with various maturities. Interest rates reflect both the probability of default and the marginal rate of substitution between present and future consumption. As in Aguiar et al. (2019), default is modeled as a stochastic outside option that can be exercised at the beginning of every period. Whenever the value of the outside option exceeds the value of repayment, the default option is triggered.

I characterize the optimal allocation by considering the modified commitment problem as the benchmark: A contract that allows the government to commit to predetermined fiscal policies but not to abstain from default. In other words, the planner simultaneously makes the fiscal decisions for all future periods and promises to pursue the plan: however, the planner cannot promise to repay debt if the outside option is preferred. The optimal allocation of the modified commitment problem defines the fiscal plan: the sequence of budget surpluses needed to repay debt, contingent on no prior default.

The first result of this paper is that the optimal allocation can be achieved under discretion. Even though the government cannot commit to future policies, it can set the term structure of its liabilities so that it has no incentive to deviate from the plan if it has an option to reoptimize the fiscal plan. The contribution of the paper is that the time-consistency result of Lucas and Stokey (1983) can be extended to environments with an opportunity of outright default.

Why might the government be willing to distort the ex ante optimal allocation in the future? At every date, the value of outstanding debt must be financed by future budget surpluses. Therefore, a deviation from the plan can be ex post beneficial if the market value of outstanding debt is decreased while the welfare is held constant. For example, if debt is mostly short term, then a reallocation of budget surpluses - which reduces the value of short-term debt at the expense of an increase in the value of long-term debt - might be optimal ex post, and vice versa. These ex-post optimal deviations are not optimal ex ante because lenders perfectly anticipate them and require higher interest rates, thus reducing the welfare of previous governments.

However, the government can structure its debt maturity such that any such distortion is not optimal in the future. The logic is similar to Lucas and Stokey (1983): the number of maturities equals the number of decisions made by the government. A contribution to Lucas and Stokey (1983) is that the number of instruments needed to ensure time consistency does not increase if the lack of commitment to repay debt is introduced to the model. The reason is that both risk-free interest rates and default premiums depend on the sequence of budget surpluses only.

The second result is that in the presence of default risk, the government issues more short-term debt than long-term debt. Moreover, the optimal maturity structure has a decaying profile: The government issues debt at all maturity dates, but the distribution of payments over time is skewed toward the short-term end. The maturity depends on the relative sensitivity of risk-free interest rates and risk premiums. The term structure is shorter if risk-free interest rates are less responsive to changes in government policies. On the other hand, if a deviation from a fiscal plan has a negligible effect on default risk, then the optimal maturity structure is approximately flat.

To gain intuition, suppose the government can manipulate both risk-free interest rates and default probabilities. Consider a deviation from ex ante optimal fiscal plan that implies reallocation of budget surpluses between two subsequent periods, keeping the welfare constant. Without loss of generality, suppose that the government reduces the budget surplus in period t and increases the budget surplus in period $t+1$. The private consumption in those periods changes as do the risk-free interest rates for bonds maturing in t and $t+1$. The default risk at period t depends on the welfare in period t: higher welfare implies lower probability. Since the perturbation implies constant welfare in period t, there is no effect on the default risk in period t. Thus, there is no change in the price of bonds maturing in t due to a change in default probabilities. However, the welfare at $t+1$ decreases because of a rise in budget surplus at $t+1$ meaning that the government is obliged to pay a larger amount of debt, therefore, implying an increase in default risk in period $t+1$. The higher default risk affects the prices of bonds maturing in period $t+1$ and all ensuing periods.

Therefore, the price of debt with longer maturity is more sensitive to potential future distortions in fiscal policy when compared to the price of debt with shorter maturity. A change in the price of debt with longer maturity reflects distortions in both the risk-free interest rate and the default premium, while the price of debt with shorter maturity varies only due to changes in the risk-free interest rate. To avoid deviations from an ex ante optimal fiscal plan, the government should leave its successor a larger stock of short-term debt and a lower stock of long-term debt. If, for example, the maturity structure is flat – meaning that the total payments due at different maturity dates are constant – then the government can benefit from policies that reduce the value of long-term debt because an increase in the value of short-term debt is smaller. In the case of only short-term debt, the government has incentives to conduct policies reducing the value of short-term debt.

The abovementioned logic is consistent with the findings of Lucas and Stokey (1983) and Aguiar et al. (2019). If government debt cannot be subject to outright default as in Lucas and Stokey (1983), then the optimal maturity structure is approximately flat because the changes in short-term and long-term risk-free rates are proportional and offset each other. In an environment in which risk-free rates are exogenous, but the default risk is positive and increasing in total debt issued, the government can manipulate the prices of bonds with all maturity dates except the price of a bond maturing today. The reason is that all government fiscal policies are conducted conditional on no defaulting in that period and, hence, the default risk of the current period cannot be altered. If any of long-term bonds are issued, the succeeding government always has an incentive to reduce their value. Thus, the government never issues or repays long-term debt.

The benefit of the approach introduced in this paper is that it allows to characterize the optimal maturity structure in an infinite-period model with all possible bond maturities. In quantitative exercises, I calibrate the model to the IMF's rule of thumb: an increase in interest rate by 4 basis points for every percentage point increase of the debt-to-GDP ratio above 60% (Alcidi and Gros, 2019). The model predictions of optimal maturity structure are broadly consistent with the data on the maturity structure of developed economies in both the decaying shape of debt payments and average maturity. I find that maturity shortens if default risk increases which is consistent with Broner, Lorenzoni, and Schmukler (2013) and Perez (2017). In addition, I show that a government chooses a decaying structure even if the initial maturity is only short or almost flat.

2. Related Literature

As already mentioned, this paper bridges the gap between two strands of the literature that study lack of commitment due to risk-free rate manipulation and risk premium manipulation in isolation. I build on the work of Aguiar et al. (2019) by introducing endogenous risk-free interest rates as in Lucas and Stokey (1983).

This paper also relates to the literature that investigates the time consistency of fiscal and monetary policy. Alvarez, Kehoe, and Neumeyer (2004) show that Ramsey policy can be made time consistent under the Friedman rule, i.e., zero nominal interest rate is optimal. Persson, Persson, and Svensson (2006) argue that time consistency can be achieved by structuring government nominal and indexed debt in an environment where positive nominal interest rates are optimal. In this paper, the focus is on the option of outright government default which is missing from the discussed studies. At the same time, a discussion on nominal debt and government's ability to inflate away debt is absent in this paper. I find that the fiscal policy is time consistent in a weaker sense, as discussed in Aguiar et al. (2019): A government follows an optimal sequence of fiscal policy decisions conditional on no prior default.

Maturity structure can be also used to hedge a government against fiscal shocks. Angeletos (2002) shows that in an environment with perfect commitment but incomplete markets, state-contingent debt can be replicated by the maturity structure of non-contingent debt providing complete insurance to the government. According to the quantitative exercises discussed in Buera and Nicolini (2004), such an insurance requires very large debt positions relative to GDP. However, Debortoli, Nunes, and Yared (2017) show that such large debt positions are not sustainable in an environment with lack of commitment as a government has an incentive to distort risk-free interest rates to alter the value of outstanding debt. Moreover, the authors find that the optimal maturity structure is approximately flat because minimizing the costs associated with the lack of commitment is quantitatively much more important than minimizing the costs associated with the lack of insurance. The latter conclusion rationalizes the focus of the paper on the commitment problem and abstraction from the hedging motive by setting deterministic fiscal shocks.

Maturity has been studied in international quantitative sovereign debt models. Aguiar and Gopinath (2006) were among the first to present a quantitative model with endogenous default decision in an environment with incomplete markets, as in the seminal paper by Eaton and Gersovitz (1981). Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012) find that exogenously lengthening debt maturity by introducing a consol bond with a decaying coupon rate improves the quantitative fit of such models. Arellano and Ramanarayanan (2012) extend their framework by allowing a sovereign to choose between consol bonds with different decaying rates, showing that average maturity shortens in an event of a crisis. Short-term debt in these models minimizes an incentive to dilute the value of longer-term debt, while long-term debt serves as a hedge against income shocks. In these models, the maturity structure of debt has a decaying profile by construction, while in my model I show that such debt structure is optimal. However, in contrast to the aforementioned studies, the role of long-term debt in this paper is to minimize risk-free interest rate distortions, while the hedging motive is absent.

Open economy and corporate finance literature often emphasizes the disciplining role of short-term debt. Jeanne (2009) demonstrates that short-term debt can incentivize a government to pursue a creditor-friendly policy as debt is rolled-over conditional on policy implementation. In Calomiris and Kahn (1991) and Diamond and Rajan (2001), short-term debt provides a creditor the option to liquidate a project. In this model, lenders are atomistic and cannot directly affect government's decisions. Instead, a government with lack of commitment uses debt maturity to discipline itself in the future.

3. Model

The economy is closed and consists of a government and a unit mass of atomistic households. The time is discrete and indexed by $t = 0, 1, 2, \ldots$

Preferences and Endowment. A representative household values private consumption and government spending:

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t\left(u(c_t)+\theta_t\omega(g_t)\right) \tag{1}$$

where u and ω are continuously differentiable, strictly increasing, concave functions and $\beta \in (0, 1)$ is the discount factor. θ_t represents the taste parameter for public spending. Larger θ implies a higher marginal utility of government expenditures and, hence, households would prefer more resources to be spent on public goods. I assume that all θ_t are deterministic and known at date 0. The government is benevolent and shares the same preferences.

There is no capital in the economy. Each period a representative household is endowed with $1-\tau$ units of consumption and the government is endowed with τ units of consumption. Every period the resource constraint is satisfied

$$c_t + g_t = 1\,\forall t \tag{2}$$

Bond Markets and Default. The government borrows from households. I assume that state-contingent bonds are not available, and the set of financial instruments is limited to discount bonds with all possible maturities. Define by b_t^{t+k} the government debt held by a household that is issued at date t and promises to pay one unit of consumption at $t+k$ and let q_t^{t+k} be the price of the bond. Denote by $\mathbf{b}_t = (b_t^{t+1}, b_t^{t+2}, \ldots)$ the vector of bond holdings issued at period t and let $\mathbf{q}_t = (q_t^{t+1}, q_t^{t+2}, \ldots)$ be the vector of corresponding bond prices. Without loss of generality, the government rebalances its portfolio each period, i.e., it buys back all its outstanding debt and issues new debt at all maturities.

The government can default on its debt. To sustain some positive debt in equilibrium I assume that default is not costless. More specifically, I follow Aguiar et al. (2019) and assume that every period the government has an outside option V_t^{def} that can be achieved upon default. V_t^{def} is drawn from continuous distribution F that has bounded support $[V_{min}; V_{max}]$. I make the following assumptions about the outside option:

Assumption 1

Outside option:

(i) $V_{max} \leq \sum_{j=t}^{\infty} \beta^j \left(u(1-\tau) + \theta_j \omega(\tau)\right) \forall t = 0, 1, \ldots$*;*

(ii) $\exists g_{min} > 0$*:* $V_{min} > \sum_{j=t}^{\infty} \beta^j \left(u(1-g_{min}) + \theta_j \omega(g_{min})\right) \forall t = 0, 1, \ldots;$

(iii) F *is strictly increasing on* (V_{min}, V_{max}) *and* $f(V_{max}) = 0$*;*

(iv) V_t^{def} *is independent across time and independent of debt portfolio.*

Restriction (i) ensures that the government will never choose an outside option if debt positions are zero. In addition, it guarantees that some positive level of debt can be sustained in equilibrium. Restriction (ii) implies that the government always defaults if the debt position is high enough and government spending is sufficiently low. Assumption (iii) allows for avoiding kinks in the pricing functions which ensures that the equilibrium can be characterized by first-order necessary conditions. The assumption of independence in (iv) is made to abstract away from using maturity structure for hedging motives.

Timing and Welfare. At the beginning of every period, the government decides whether or not to default on its debt. Upon default it receives the outside option value V_t^{def}. Otherwise, the government sets public spending, buys back existing debt and issues new debt. The default decision precedes any fiscal decisions, and the government is not allowed to default until the beginning of the next period once new debt has been issued. This timing rules out the possibility of self-fulfilling debt crises discussed by Cole and Kehoe (2000).

To simplify notation, it is useful to define the contingent budget surplus as the difference between endowment of and spending by the government if it does not default:

$$s_t = \tau - g_t.$$

Consumption is then defined as $c_t = 1 - \tau + s_t$ and government spending is $g_t = \tau - s_t$. In addition, let $u_t = u(1 - \tau + s_t)$, $\omega_t = \omega(\tau - s_t)$, $u'_t = u'(1 - \tau + s_t)$, $\omega'_t = \omega'(\tau - s_t)$ and $u''_t = u''(1 - \tau + s_t)$.

Define by $\mathbf{s}_t = (s_t, s_{t+1}, \dots)$ a sequence of contingent budget surpluses. Then prior to a default decision, the welfare can be defined recursively as follows:

$$W_t(\mathbf{s}_t) = u_t + \theta_t \omega_t + \beta \cdot \mathbb{E}\max\{W_{t+1}(\mathbf{s}_{t+1}), V_{t+1}^{def}\}. \tag{3}$$

In any competitive equilibrium, household optimality conditions must be satisfied. A representative household takes into account the current and future government policies that are reflected in risk-free interest rates and risk premiums. The price of a bond that matures in $k \geq 1$ periods is derived from household optimality conditions and equals

$$q_t^{t+k} = \beta^k \cdot \frac{u'_{t+k}}{u'_t} \cdot Pr_t^{t+k} \tag{4}$$

where

$$Pr_t^{t+k} = \prod_{i=t+1}^{t+k} F\left(W_i(\mathbf{s}_i)\right) \tag{5}$$

denotes the probability of no default from period $t+1$ to $t+k$. Note that this probability is a function of $\mathbf{s}_{t+1}$ but does not depend on s_k, $k \leq t$.

Conditional on no prior default the budget constraint in every period satisfies:

$$s_t + \mathbf{q}_t \cdot \mathbf{b}_t \geq (\mathbf{1}, \mathbf{q}_t) \cdot \mathbf{b}_{t-1}. \tag{6}$$

The right-hand side of (6) is the market value of outstanding debt. The left-hand side is the sum of the budget surplus and the market value of newly issued debt.

4. Modified Commitment Problem

Consider a planner who can commit to fiscal policies but cannot commit to pay its debt. In other words, at date 0 a planner simultaneously makes fiscal decisions for all periods, and it can promise to follow the plan; however, the planner defaults whenever the value of the outside option is higher than the value of pursuing the fiscal plan. I call this the modified commitment problem.

At period 0 the planner inherits the outstanding debt $\mathbf{b}_{-1}$ and, given default decision is not optimal, chooses a fiscal plan $\mathbf{s}_0$ to maximize the welfare (3) for $t = 0$. The optimal fiscal plan must satisfy the dynamic budget constraint

$$s_0 + \sum_{t=1}^{\infty} \beta^t \cdot \frac{u_t'}{u_0'} \cdot Pr_0^t \cdot s_t \geq b_{-1}^0 + \sum_{t=1}^{\infty} \beta^t \cdot \frac{u_t'}{u_0'} \cdot Pr_0^t \cdot b_{-1}^t$$

or equivalently

$$\sum_{t=0}^{\infty} \beta^t \cdot u_t' \cdot Pr_0^t \cdot s_t \geq \sum_{t=0}^{\infty} \beta^t \cdot u_t' \cdot Pr_0^t \cdot b_{-1}^t \tag{7}$$

with $Pr_0^0 \equiv 1$. The left-hand side of the constraint represents the present value of contingent budget surpluses, while the right-hand side is the market value of outstanding debt (both the left-hand side and right-hand side are adjusted by u_0'). Loosely speaking, any outstanding debt must be financed by future budget surpluses.

To simplify the notation, let

$$S_t = \sum_{k=0}^{\infty} \beta^k \cdot u_{t+k}' \cdot Pr_t^{t+k} \cdot s_{t+k} = u_t' \cdot s_t + \beta \cdot Pr_t^{t+1} \cdot S_{t+1}, \tag{8}$$

$$B_{-1}^t = \sum_{k=0}^{\infty} \beta^k \cdot u_{t+k}' \cdot Pr_t^{t+k} \cdot b_{-1}^{t+k} = u_t' \cdot b_{-1}^t + \beta \cdot Pr_t^{t+1} \cdot B_{-1}^{t+1}. \tag{9}$$

Then S_0 corresponds to the market value of contingent budget surpluses (adjusted by u_0') - the left-hand side of (7). Similarly, B_{-1}^0 is the market value of outstanding debt $\mathbf{b}_{-1}$ at date 0 (adjusted by u_0') if the planner pursues fiscal plan $\mathbf{s}_0$ - the right-hand side of (7). In general, S_t shows the present value of the stream of contingent budget surpluses s_t, s_{t+1}, … from the perspective of period t. Analogously, B_{-1}^t shows the market value of debt issued in period -1 and maturing in period t or later: b_{-1}^t, b_{-1}^{t+1}, …. Note that if $s_t = s_{t+1}$ $\forall t$, S_t is constant. Similarly, if initial maturity structure is flat then B_{-1}^t also remains unchanged for all t.

First-Order Optimality Condition

Optimal fiscal plan $\mathbf{s}_0(\mathbf{b}_{-1})$ satisfies the first-order necessary conditions of the modified commitment problem:

$$\frac{\frac{\partial}{\partial s_{t+1}} W_0(\mathbf{s}_0)}{\frac{\partial}{\partial s_t} W_0(\mathbf{s}_0)} = \frac{\frac{\partial}{\partial s_{t+1}} (S_0 - B_{-1}^0)}{\frac{\partial}{\partial s_t} (S_0 - B_{-1}^0)}. \tag{10}$$

The left-hand side of (10) is the marginal rate of substitution between contingent budget surpluses at period t and $t+1$. The marginal rate of substitution shows by how much the budget surplus at period t can be decreased if the planner increases the budget surplus at $t+1$ by one unit, keeping planner's welfare (3) constant. Note that the marginal rate of substitution depends only on fiscal plan $\mathbf{s}_0$ and does not depend on the initial debt composition $\mathbf{b}_{-1}$. The right-hand side of the equation (10) shows the rate at which the planner at period 0 can transfer budget surpluses from $t+1$ to t, keeping the budget constraint satisfied with equality. In equilibrium, any marginal deviation from the optimal fiscal plan which satisfies the budget constraint (7) does not lead to an increase in the planner's welfare.

Lemma 1. The Optimality Condition

The first-order necessary condition (10) can be simplified as follows:

$$\frac{\theta_{t+1}\omega'_{t+1} - u'_{t+1}}{\theta_t\omega'_t - u'_t} = \frac{\dfrac{\partial(u'_{t+1}\cdot(s_{t+1} - b_{-1}^{t+1}))}{\partial s_{t+1}} + \dfrac{\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}}}{Pr_t^{t+1}}(S_{t+1} - B_{-1}^{t+1})}{\dfrac{\partial(u'_t\cdot(s_t - b_{-1}^t))}{\partial s_t}}. \tag{11}$$

To get intuition, consider the following perturbation: suppose the government increases s_t and decreases s_{t+1} keeping $W_0(\mathbf{s}_0)$ constant. Note that $W_0(\mathbf{s}_0)$ remains constant only if $W_t(\mathbf{s}_t)$ does not change. An increase in s_t decreases $W_t(\mathbf{s}_t)$ by $(\theta_t\omega'_t - u'_t)\triangle s_t$. A reduction in s_{t+1} has several effects. First, it increases $W_{t+1}(\mathbf{s}_{t+1})$ by $(\theta_{t+1}\omega'_{t+1} - u'_{t+1})\triangle s_{t+1}$. Second, as default is triggered if $V_{t+1}^{def} > W_{t+1}(\mathbf{s}_{t+1})$ and $W_{t+1}(\mathbf{s}_{t+1})$ goes up, the probability of receiving the outside option V_{t+1}^{def} decreases while the probability of repaying debt and receiving $W_{t+1}(\mathbf{s}_{t+1})$ increases. However, for a small $\triangle s_{t+1}$ the second effect disappears because an increase in $\mathbb{E}\max\{W_{t+1}(\mathbf{s}_{t+1}), V_{t+1}^{def}\}$ due to an increase in the probability of $W_{t+1}(\mathbf{s}_{t+1})$ is offset by a decrease in the expected value of V_{t+1}^{def}[2]. Finally, the ratio of $\triangle s_t$ and $-\triangle s_{t+1}$ is proportional to $\theta_t\omega'_t - u'_t$ and $\beta Pr_t^{t+1}(\theta_{t+1}\omega'_{t+1} - u'_{t+1})$. The term βPr_t^{t+1} disappears as it cancels out with the right-hand side discussed below.

Now consider the right-hand side of (11). The optimality of the fiscal plan requires that such perturbation does not allow the planner to relax the budget constraint (7). Keeping $W_t(\mathbf{s}_t)$ constant, the perturbation in s_t and s_{t+1} affects the budget constraint via three channels. First, the present value of budget surpluses changes as a result of changes in s_t and s_{t+1}. The second channel is through changes in risk-free interest rates as u'_t and u'_{t+1} change. Note that this channel alters the present value of budget surpluses in periods t and $t+1$ and bonds maturing in t and $t+1$ only. These two channels reflect the

[2] See Lemma A.1 in Appendix for a formal proof.

denominator and the first term of numerator of the right-hand side in (11). Third, the perturbation distorts default probabilities. However, as $W_k(\boldsymbol{s}_k)$ changes only for $k = t+1$ and remains constant for all other periods, the perturbation affects only Pr_t^{t+1} - the default risk in period $t+1$. The last channel affects the present value of budget surpluses in periods $t+1$ and onwards, as well as all outstanding debt maturing in $t+1$ and later. This change is reflected in the second term of the numerator of the right-hand side of the optimality condition.

Role of Maturity Structure

Note that only the maturity structure of outstanding debt is important for the planning problem. The maturity structure of debt issued by the planner in any subsequent period is completely irrelevant. As long as the planner can commit to the sequence of budget surpluses, default probabilities and risk-free interest rates remain constant. Therefore, bond prices do not change and there are infinitely many ways to implement the allocation, with multiple maturities available every period.

However, the irrelevance of maturity does not apply to a government with lack of commitment. Suppose that a government can reoptimize in the future by choosing a new fiscal plan. Similarly to the period 0 planning problem, the decision of such a government depends on the maturity structure of outstanding debt. Therefore, the preceding government chooses the term structure of issued debt strategically to affect the decisions of the future governments. In the next section I show that there is a unique maturity structure that induces the governments to follow the ex ante optimal fiscal plan if they have discretion.

5. Time Consistency Under Discretion

Suppose that the planner designs an optimal fiscal plan $\mathbf{s}_0^\star(\mathbf{b}_{-1})$, and given the realization of $\{V_t^{def}\}_{t=1,2,\ldots,T}$, no default decision is triggered till period $T > 0$. Then suppose that in period T the planner receives an unexpected option to redesign the existing fiscal plan. Denote by $\mathbf{s}_T^\star(\mathbf{b}_{T-1})$ the fiscal plan chosen by the planner in period T which is the function of outstanding debt portfolio $\mathbf{b}_{T-1}$. For any $t \geq T$ the optimal fiscal plan $\mathbf{s}_T^\star(\mathbf{b}_{T-1})$ satisfies the optimality conditions:

$$\frac{\theta_{t+1}\omega'_{t+1} - u'_{t+1}}{\theta_t \omega'_t - u'_t} = \frac{\dfrac{\partial\left(u'_{t+1}\cdot\left(s_{t+1} - b_{T-1}^{t+1}\right)\right)}{\partial s_{t+1}} + \dfrac{\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}}}{Pr_t^{t+1}}\left(S_{t+1} - B_{T-1}^{t+1}\right)}{\dfrac{\partial\left(u'_t\cdot\left(s_t - b_{T-1}^{t}\right)\right)}{\partial s_t}}. \tag{12}$$

Note that the optimality conditions (11) and (12) are quite similar. The left-hand sides of the equations – the marginal rates of substitution between contingent budget surpluses in period t and $t+1$ – are identical for both planners in period 0 and period T. The reason is that the preferences of a government do not change over time in this model. However, the right-hand sides of (11) and (12) – showing a rate at which a planner can reallocate budget surpluses over time while keeping the budget constraint satisfied – are different. In both cases, the planners choose optimal fiscal plans taking into account how perturbations in budget surpluses s_t and s_{t+1} affect the market value of outstanding debt through changes in risk-free interest rates and default risks. Therefore, given that generally $b_{-1}^{t} \neq b_{T-1}^{t}$, ex ante optimal fiscal plan $\mathbf{s}_0^{\star}(\mathbf{b}_{-1})$ and ex post optimal fiscal plan $\mathbf{s}_T^{\star}(\mathbf{b}_{T-1})$ are not identical for any maturity structure.

Nevertheless, there is a generally unique maturity structure which makes the solution to be time consistent. The reason is that the government at $T-1$ has enough instruments to influence the successor's optimization problem. Let $\mathbb{T} \to \infty$ be the total number of periods. The fiscal plan of the planner in period T consists of $\mathbb{T}-T+1$ elements $(s_T, s_{T+1}, \ldots, s_{\mathbb{T}})$ which is the solution to $\mathbb{T}-T$ optimality conditions ([eq:FOCT]) and the dynamic budget constraint. The planner at $T-1$ has exactly $\mathbb{T}-T+1$ maturities $(b_{T-1}^{T}, b_{T-1}^{T+1}, \ldots, b_{T-1}^{\mathbb{T}})$ to affect the fiscal plan $\mathbf{s}_T = (s_T, s_{T+1}, \ldots, s_{\mathbb{T}})$. In addition, note that combining endogenous risk-free interest rates and endogenous default premiums does not require more instruments because both risk-free interest rates and default risks are functions of the sequence of contingent budget surpluses only and do not depend on outstanding debt.

Lemma 2. Optimal Maturity Structure

The ex ante optimal fiscal plan $\mathbf{s}_0^{\star}(\mathbf{b}_{-1})$ satisfies the optimality condition (12) of a planner in period T if the maturity structure of outstanding debt $\mathbf{b}_{T-1}$ satisfies:

$$\frac{\theta_{t+1}\omega_{t+1}' - u_{t+1}'}{\theta_t\omega_t' - u_t'} = \frac{\dfrac{\partial(u_{t+1}' \cdot (b_{T-1}^{t+1} - b_{-1}^{t+1}))}{\partial s_{t+1}} + \dfrac{\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}}}{Pr_t^{t+1}}(B_{T-1}^{t+1} - B_{-1}^{t+1})}{\dfrac{\partial(u_t' \cdot (b_{T-1}^{t} - b_{-1}^{t}))}{\partial s_t}}. \tag{13}$$

To gain intuition, recall that all the changes in the stocks of debt are optimal if the planner at T has no incentive to deviate from the ex ante optimal allocation. The latter is possible only if the market value of the new debt issued cannot be affected by perturbations in s_t and s_{t+1} while keeping the welfare constant. Consider an increase in s_t and a decrease in s_{t+1} that keep W_t constant. These changes affect the marginal utility of consumption in periods t and $t+1$ and welfare in period $t+1$, W_{t+1}, distorting the default risk in period $t+1$. The changes in marginal utilities affect the market value of debt maturing at t and $t+1$ due to changes in risk-free interest rates. The change of the default risk in $t+1$ affects the market value of debt maturing in $t+1$ and all consequent periods.

In addition, note that the changes in the market value of the long-term debt (maturing in $t+1$ and consequent periods) due to distortions in risk-free interest rates and default risks have the same sign. For example, if s_{t+1} decreases, the marginal utility of consumption in $t+1$ leading to an increase in the bond price due to the changes in risk-free rates. The welfare at $t+1$ also increases, thus, the price of a bond raises further as default risk drops. Thus, generally there is an asymmetry between shorter-term debt and longer-term debt because the market value of shorter-term debt is less sensitive to potential deviations of future governments from an ex ante optimal fiscal plan.

6. Decaying Maturity Profile

In this section I analytically prove that in the presence of default risk the maturity structure has a decaying profile implying that a government issues and keeps larger stock of debt with shorter maturity. Let us make the following assumption:

Assumption 2

(i) $\theta_0 > \theta_1 = \theta_2 = \ldots = 1$;

(ii) $\omega'(\tau) \geq u'(1-\tau)$;

(iii) $u(c) = \frac{c^{1-\gamma_C}}{1-\gamma_C}$, $\gamma_C \geq 0$.

Assumption (i) states that there is an incentive to have a higher public spending in the very first period than in the consecutive periods. Assumption (ii) states that if the budget surplus is non-negative, then the marginal utility of public spending weekly exceeds the marginal utility of private consumption. Together these two assumptions ensure that the government has an incentive to reallocate resources from future periods to the initial period by issuing some debt.

It is useful to define $\bar{s} > 0$ as the maximum stationary budget surplus for which the default risk remains zero every period:

$$\frac{u(1-\tau+\bar{s}) + \omega(\tau-\bar{s})}{1-\beta} = V_{max}.$$

Suppose that the initial debt is zero. Note that the optimality condition (11) is satisfied for all $s_1 = s_2 = \ldots = \bar{s}$ because $\theta_1 = \theta_2 = \ldots = 1$. Let $\bar{s}_0 < 0$ be the budget surplus such that the dynamic budget constraint (7) is satisfied. As the left-hand side of (11) is strictly decreasing in θ_t, we conclude that there exists a unique $\bar{\theta} > 1$ which satisfies the optimality condition (11) for $t = 0$, $s_0 = \bar{s}_0$ and $s_1 = s_2 = \ldots = \bar{s}$.

Before proceeding with the profile of the optimal maturity structure, let us qualitatively characterize the optimal fiscal plan.

Proposition 1. The Optimal Fiscal Plan

Suppose there is no initial debt. Then:

(i) for $\theta_0 \in [1, \bar{\theta}]$, $s_1 = s_2 = \ldots \leq \bar{s}$ *and* $Pr_t^{t+1} = 1\ \forall t \geq 1$*;*

(ii) for $\theta_0 > \bar{\theta}$, $s_1 > s_2 > \ldots > \bar{s}$ *and* $Pr_t^{t+1} < 1\ \forall t \geq 1$*.*

According to Proposition 1, if $\theta_0 \leq \bar{\theta}$ the planner prefers to stay in the "safe" region in which default risk is zero in every period. If $\theta_0 > \bar{\theta}$ then the planner enters the "crisis" region in which default risk is always positive.

In the safe region, the only motive for the planner is to smoothen public spending over time. If the planner deviates from fixed budget surpluses by marginally increasing the budget surplus in one period by $\beta \triangle s$ and marginally reducing the budget surplus in the next period by $\triangle s$, the planner's welfare strictly decreases because the per-period utility is strictly concave. Moreover, this deviation strictly decreases the present value of budget surpluses because interest rates move against changes in budget surpluses. The opposite deviation – a decrease in budget surplus in the first period and an increase in the second – has exactly the same implication.

In the crisis region, there is also the saving motive to reduce default risk. If a planner marginally increases s_t and marginally decreases s_{t+1}, the effect on $W_t(\mathbf{s}_t)$ is negligible as it depends on both s_t and s_{t+1}. However, $W_{t+1}(\mathbf{s}_{t+1})$ increases as it depends on s_{t+1} but not s_t. This increase reduces the default risk in period $t+1$ and, thus, relaxes the dynamic budget constraint. However, the opposite perturbation of budget surpluses has the opposite adverse effect on the default risk. Therefore, the optimal fiscal policy is to pay off a larger fraction of debt in earlier periods to benefit from lower long-term risk of default.

Next I turn to the analysis of the shape of the maturity structure. We know from the literature that in an environment with no default the optimal maturity structure is approximately flat (see Lucas and Stokey (1983), Debortoli, Nunes, and Yared (2017)). Alternatively, if we consider a model with an opportunity to default but lenders are risk neutral, then the optimal debt policy is to issue short-term debt only (see Aguiar et al. (2019)). The abovementioned findings are also the solutions to this model.

Therefore, an interesting case is the profile of the maturity structure if borrowers are risk-averse and default risk is present. Let's suppose that the initial debt is zero to abstract away from the potential effect of the initial debt on the optimal fiscal plan and debt management. Consider $\theta_0 > \bar{\theta}$ so that the economy is in the crisis region with a strictly positive probability of default. According to Proposition 2, the maturity structure has a decaying profile.

Proposition 2. Decaying profile of the Maturity Structure

Suppose that initial debt is zero and $\theta_0 > \bar{\theta}$. Then the optimal maturity structure has a decaying profile:

$$b_T^{T+1} > b_T^{T+2} >\ldots> 0 \ \forall T \geq 0.$$

The main reason for the decaying profile of debt maturity is the asymmetry of the responses of short-term and long-term interest rates to perturbations in fiscal plans. Consider again a decrease in s_t and an increase in s_{t+1} while keeping welfare constant. The optimality conditions require this deviation from the fiscal plan to have no effect on the market value of debt. The perturbation affects the marginal utility of consumption at t and $t+1$, thus, the market value of debt maturing at t and $t+1$ alters due to changes in risk-free interest rates. In addition, the perturbation affects default risk in period $t+1$ as the welfare W_{t+1} decreases due to an increase in s_{t+1}. Note that the default risk in period t remains unchanged: the perturbation does not affect initial welfare W_0 so that a decrease in s_t is offset by an increase in s_{t+1}. As a consequence, the market value of long-term debt maturing in $t+1$, $t+2$ and so on is affected by the increase in default risk in period $t+1$. Meanwhile, the market value of short-term debt maturing in t and before is not affected by changes in default probabilities. Another reason is the decreasing stream of budget surpluses. As private consumption is higher compared to the private consumption in the subsequent period, risk-free interest rates are more sensitive to changes in consumption in later periods.

To sum it up, the long-term interest rates are more elastic and can be more easily manipulated by a government. A perturbation in fiscal policy leads to more substantial changes in the market value of long-term debt than the short-term debt due to more responsive risk-free interest rates and default risks. Therefore, the government issues more short-term debt and less long-term debt so that the changes in the market value of short-term debt and long-term debt cancel each other.

An important observation is that the maturity structure does not (directly) depend on the levels of risk-free interest rates and default risks. The term structure of debt rather depends on the responsiveness of risk-free interest rates and default probabilities to marginal changes in fiscal policy. Indeed, consider an economy with some constant risk of default that does not depend on the level or term structure of debt[3]. Suppose that the planner optimally decides to finance initial budget deficit with equal budget surpluses every period. Then, according to equation (13), the term structure of debt would be exactly flat violating the conclusion of Proposition 2. Hence, it is not the presence of default risk which skews the maturity profile toward the short end, but the continuous dependence of default risk on fiscal policies conducted by the government.

[3] A discrete distribution of the outside option value can lead to (locally) constant default probabilities.

7. Quantitative Exercises

In this section, I numerically solve for the optimal debt policy. Then the maturity is compared with empirical observations to show that it resembles a bond with a decaying coupon. In addition, I discuss the impact of initial maturity structure on the maturity structure of issued debt.

Functional Forms and Calibration

Throughout this section I assume that – conditional on no prior default – the government's per-period payoff is:

$$\frac{c^{1-\gamma_C}-1}{1-\gamma_C}+\kappa\frac{g^{1-\gamma_G}-1}{1-\gamma_G}.$$

The initial debt is assumed to replicate a bond with a decaying coupon that pays interest rate $r = 1/\beta - 1$. Each coupon decreases at a constant rate δ. The debt distribution over time is given by

$$b_{-1}^{t} = b\cdot(1-\delta)^{t}\cdot(\delta+r) \tag{14}$$

where b is the debt-to-GDP ratio. The maturity of initial debt is:

$$m=\frac{1+r}{\delta+r}.$$

I set $\beta = 0.96$ to match the annual risk-free interest rate at approximately 4%. The relative risk aversion coefficients of households γ_C and government γ_G are set to 2, which is consistent with sovereign debt literature. Lump-sum taxes τ are set at 0.31 so that the ratio of government spending to private consumption equals 40% in an economy with a 60% debt-to-GDP ratio. The taste parameter for government spending κ equals 0.20 so that $\omega'(\tau) = u'(1-\tau)$. The decaying coupon rate δ is fixed at 0.107 to match the maturity of initial debt at seven years.

The distribution of the outside option V_t^{def} is:

$$f(v)=\begin{cases}\alpha_0+\alpha_1 v & v\in(V_{min},V_{max}),\\ 0 & \text{otherwise.}\end{cases}$$

Note that it is sufficient to calibrate only two distribution parameters out of four (α_0, α_1, V_{min}, V_{max}) . The other two are given by the property $\int_{V_{min}}^{V_{max}} f(v)dv = 1$ and Assumption 1 (iii) $f(V_{max}) = 0$. Thus, only two data moments are required to calibrate the distribution of the outside option.

I set the distribution to match the causal effect of debt to GDP on interest rates. According to Alcidi and Gros (2019), the IMF's simple rule of thumb states that the interest rises by 4 basis points for every one percentage point increase in the debt-to-GDP ratio above 60%. This rule is confirmed by a number of empirical studies. The non-linear nature of this relationship is highlighted by Ardagna, Caselli, and Lane (2007). They find that debt to GDP has a positive effect on interest rates if the ratio is above 62.5-65.4%. Elasticity of about 2-4 points is found in Engen and Hubbard (2005), Gruber and Kamin (2012), and Laubach (2009). Thus, the targeted moments are: (i) 60% is the maximum riskless debt-to-GDP ratio; (ii) at 100% debt to GDP, an increase in the debt-to-GDP ratio by one percent increases the short-term default risk by four basis points.

While the first moment follows directly from the IMF's rule of thumb, the second moment requires several clarifications. First, as the elasticity in this model is not constant and increases with the debt-to-GDP ratio, the elasticity is set to 4 at the average level of debt-to-GDP ratio of developed countries[4]. Second, this paper focuses on the effect of the risk premium only as changes in risk-free interest rates are minor. Finally, the focus is on short-term interest rates rather than long-term interest rates. One of the properties of this model is that debt to GDP decreases over time if the default risk is positive. Thus, the long-term default risk is lower than the short-term risk. Figure 1 shows the predicted probability of default in the next period as a function of debt to GDP.

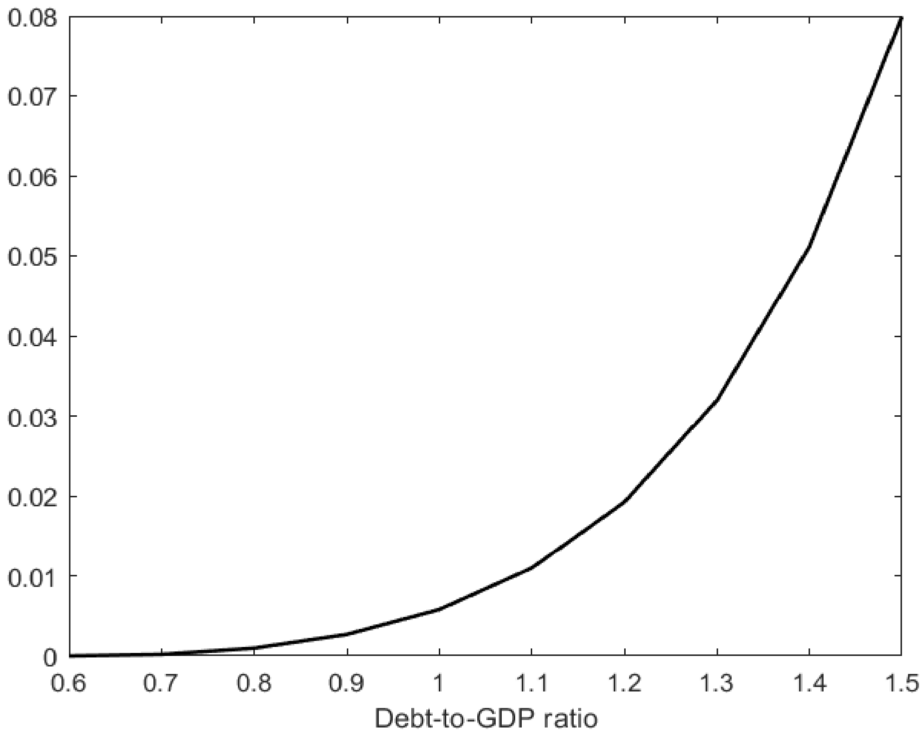

Figure 1. Default risk as a function of debt to GDP.

[4] according to the IMF WEO (October 2019), the average debt to GDP for advanced economies is 103.8%

The computational algorithm consists of two major steps. The first step is to solve for the optimal fiscal plan which is defined is Section 3. The procedure is start by guessing s_0 and solving for s_1, s_2, …, s_T (where T is a large number) given the optimality condition (11) and the budget constraint holds with strict inequality. The assumption is that in the long-run the difference between present value of budget surpluses and outstanding debt vanishes. If the initial value of s_0 is too small, then the sequence of budget surpluses is insufficient to repay outstanding debt. Otherwise, if the initial value of s_0 is too large, the government repays its debt and accumulates positive assets. The second step is solving for the optimal maturity structure. The optimal maturity structure satisfies condition (13) and the budget constraint (6).

Fiscal Policy and Maturity Profile in a Benchmark Case

Figure 2 presents the optimal fiscal and debt policy if initial debt is equal to 100% of GDP. The top right panel exhibits the maturity structure of outstanding debt. The top left panel displays a decreasing sequence of contingent budget surpluses. The profile of budget surpluses is much flatter in comparison to the maturity structure of debt. The decaying pattern of fiscal policy explains slightly increasing risk-free interest rates and a decreasing default risk displayed in the two bottom panels. Overall, the yield curve has a slightly negative slope. The maturity of net debt issued equals 7.47 years which is longer than the maturity of initial debt. The shape of the newly issued debt seems to be very similar to the initial debt maturity profile.

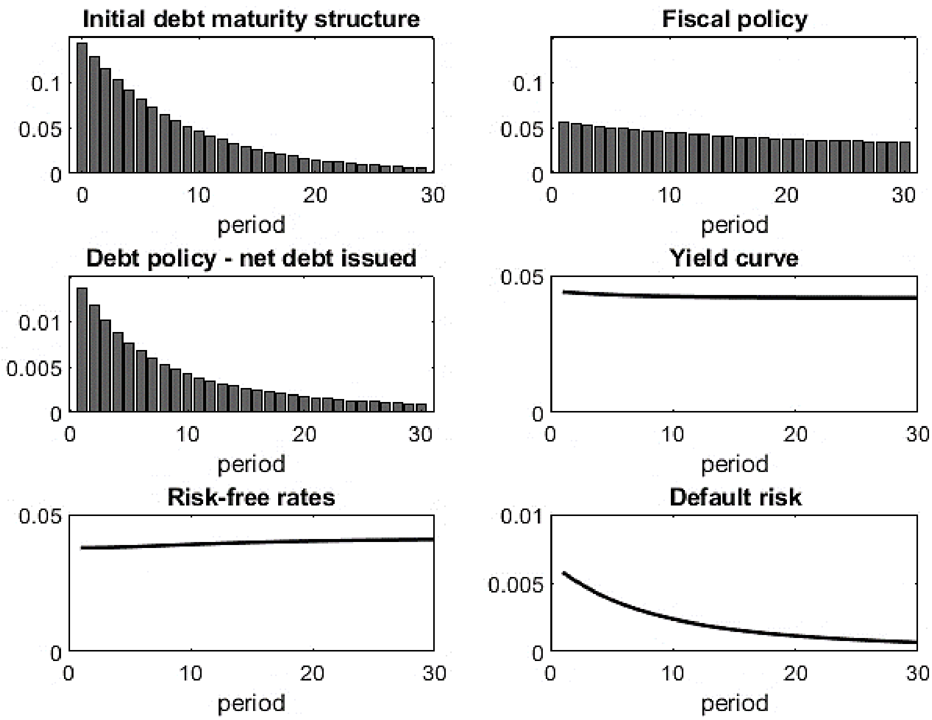

Figure 2. Benchmark case.

Figure 3 depicts the structure of debt issued by the government and its approximation with a decaying coupon bond. The decaying coupon bond has exactly the same maturity and net present value of debt. The optimal debt policy prescribes issuing more of one-, two- and three-period debt compared to the bond approximation. Then the payments from period 5 till period 25 are lower followed by higher payments in the consequent years. In other words, the decaying rate of the optimal debt issuance is not constant and the profile is relatively steeper for the short-term debt and flatter for the long-term debt.

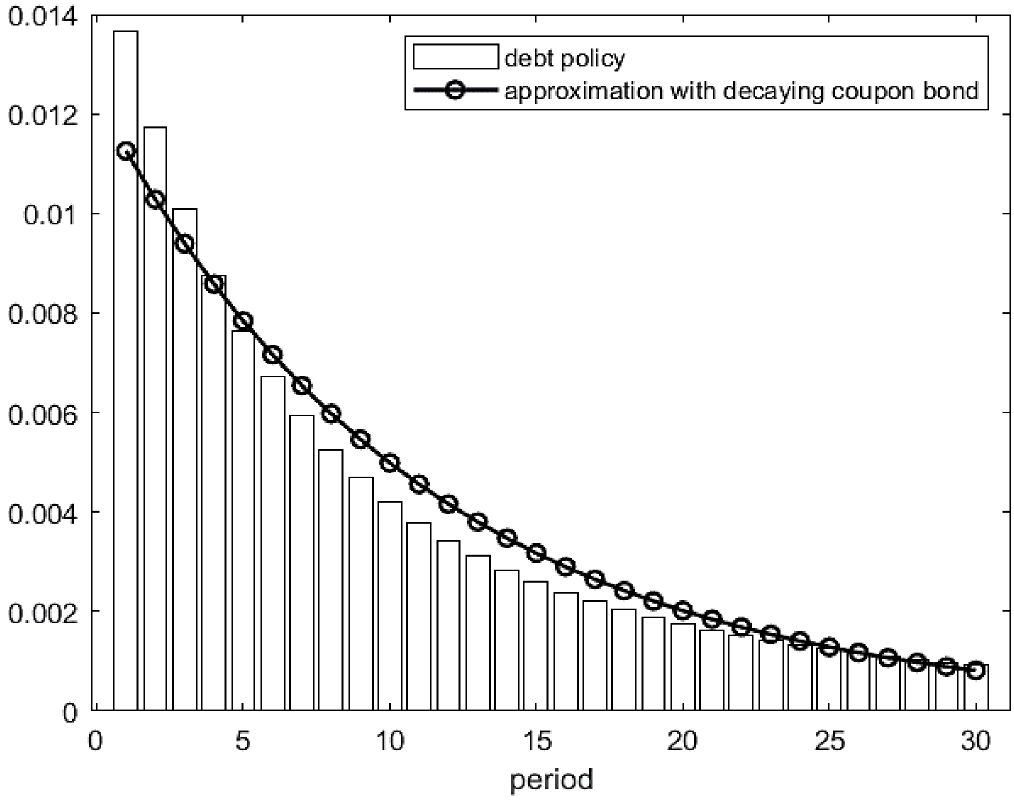

Figure 3. Approximation of the optimal maturity structure with a decaying coupon bond.

Initial Maturity Structure

In this subsection I demonstrate that the decaying payment scheme is preserved even if the maturity profile of initial debt is not decaying. I consider two extreme cases: one-period debt only and flat maturity structure. These two cases correspond to $\delta = 1$ and $\delta = 0$. The optimal debt policy is presented on Figure 4. The top panel corresponds to short initial debt. As in the benchmark model, a government issues bonds with all maturities and the debt structure has a decaying shape.

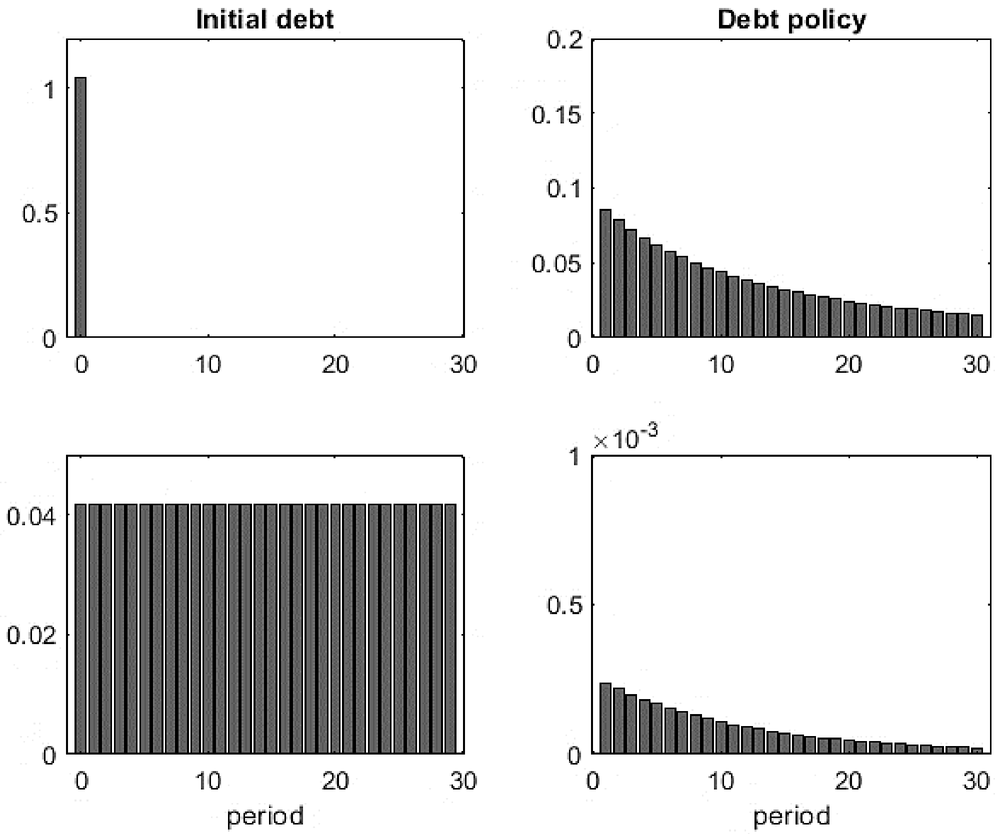

Figure 4. Short and long initial debt.

The bottom panel presents the optimal debt policy if initial maturity is flat. Particularly, the case is considered when a government owes constant payments for the next $T = 100$ years. The amount of debt issued by the government is tiny compared to the total stock of debt. Still, the decaying profile is preserved.

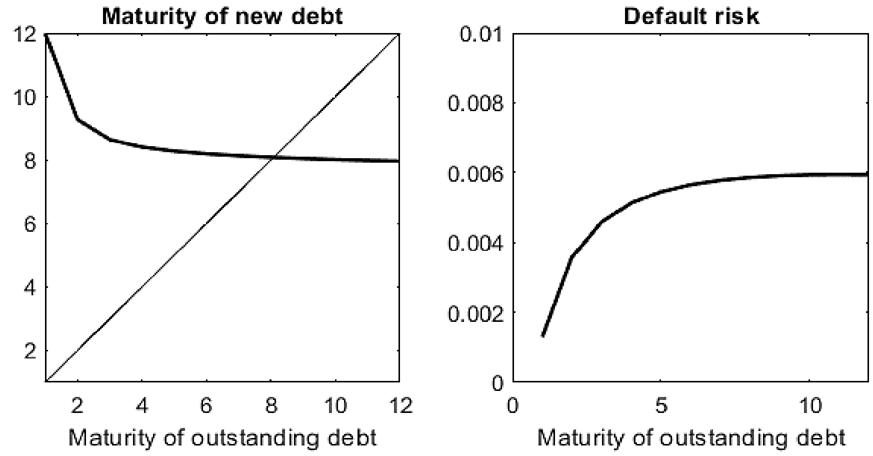

Figure 5. Maturity structure as a function of initial maturity.

Finally, this paper studies how initial maturity affects the maturity of issued debt. The left panel of Figure 5 depicts the relationship between the maturity of outstanding debt and the maturity of issued debt. In this exercise, the debt-to-GDP ratio stays constant at 100% and the payment scheme satisfies (14). The only parameter that varies is the decay rate δ. As the maturity of initial debt increases, the maturity of issued debt gets shorter with the fixed point slightly above eight years. The main reason for the inverse relationship is the increasing default risk presented on the right panel of Figure 5. If initial debt is shorter, then a larger stock of debt must be repaid or rolled over. As the price of debt is decreasing in its amount issued due to both endogenous risk-free interest rates and risk premium, a government chooses a larger budget surplus if outstanding short-term debt is larger. The latter reduces the government's liabilities and default risk in the following periods. A smaller default probability in turn corresponds to a lower elasticity of default risk, implying that the issued debt is flatter.

The Maturity Structure of Developed Countries

Figure 6 displays the maturity structure of marketable bonds for the following countries: the U.S., Japan, Germany, France, Italy and the U.K. The U.S. debt is in millions USD. The debts of the German, Italian and French governments are in millions EUR. The U.K. debt is in millions GBP and the Japanese debt is in millions JPY. The data was collected on 17th of October, 2017 and each bar shows the principal and interest payments owed by a government that must be paid by the government in a given year as of October 17, 2017. Therefore, the payments due in the end of October, November and December 2017 are skipped and the presented date starts in 2018. In each panel the first bar represents the total payments owed by a government due in 2018, the second bar represents the total payments due in 2019 and so on. The only exception is the very last bar in each panel that includes payments due in 2047 and all future years. The last bar is somewhat higher for Italy, Japan and France, but most importantly it represents a considerable part of debt for the United Kingdom. The main reason is that the British Government actively issued consol bonds during the Industrial Revolution (see Mokyr (2010)). Due to this aggregation of debt, the last bar in the discussion of maturity data is ignored.

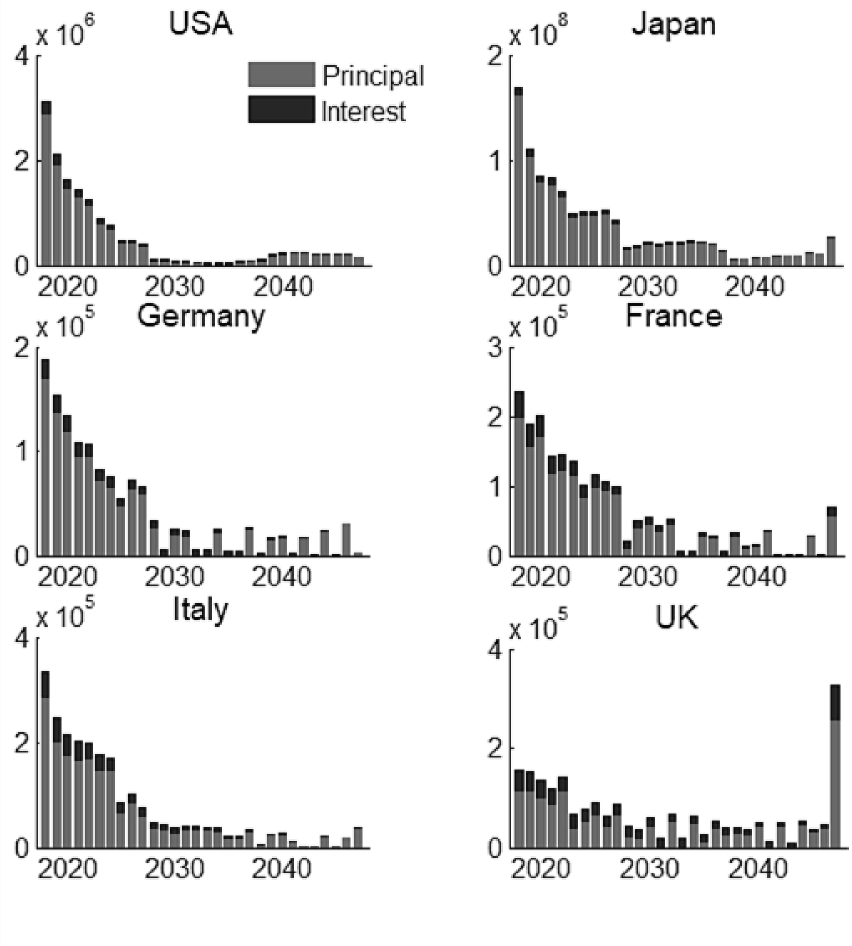

Figure 6. Maturity structure of the USA, Japan, Germany, France, Italy and the UK.

Source: Bloomberg.

The maturity statistics are broadly consistent with the predictions of the model. First of all, debt is skewed toward the short-term end. Even though the countries issue bonds maturing in 30 years and later, the average maturity for the US, Japan, Germany, France and Italy is 5.79, 7.74, 6.8, 7.83 and 6.8 years respectively. It is worth noting that the maturity structure of the UK government debt is much flatter, and the average maturity is 14.97 years. For each country the stock of debt maturing in one year is the largest[5]. In 2018 the US government has to pay (or roll-over) more than 20% of the total debt. The debt to be paid by the US government in the next five years constitutes 62% of the total debt. Similarly, total amount of debt maturing in the next five years constitutes approximately 50% of the total debt of Japan, Germany, France and Italy[6].

Moreover, the maturity structure has a decaying profile as predicted by the model. The U.S. maturity structure exhibits the decaying profile for 15 years: payments due in 2018-2033 are strictly decreasing. Then the term structure does display some increasing trend. However, note that the total debt due in 2034 and all later years is lower than the debt due

[5] Recall the last bar that aggregates total payments due in 2047 and later years is ignored.
[6] It is 48%, 53%, 46% and 52% for Japan, Germany, France and Italy, respectively.

in 2018. Debt term structures of Japan, Germany, France and Italy have similar patterns. Even much flatter U.K. debt tends to decline over maturity date: the total debt maturing in 1-5 years amounts to approximately 32%, while the total debt to be paid in 6-10 years constitutes less than 18%.

One of the reasons the maturity structure might be not perfectly decaying is that the number of issuances is limited and most of them are short term. For example, the number of French debt active issuances is only 95[7]. There are no principal payments due in 2033, 2034, 2037 and some subsequent years. The number of issuances can be limited due to some fixed costs or other frictions.

8. Summary

This paper shows that in an environment with endogenous risk-free interest rates and endogenous default premiums the optimal maturity structure has a decaying profile. An important assumption is that marginal changes in fiscal policies lead to a marginal change in the risk-free interest rate and the default risk. Moreover, in this model fiscal policy is time consistent. Therefore, the time-consistency result discussed in Lucas and Stokey (1983) remains intact if an opportunity of outright default is considered. The model enables analyzing the maturity profile of sovereign debt with an arbitrarily large number of maturities. The numerical exercises show consistency of model predictions with empirical evidence.

The skewness of the debt profile is the result of the asymmetry in the elasticity of short-term and long-term interest rates. The long-term interest rates are more responsive to perturbations in fiscal policies due to higher sensitivity of long-term default risk. In addition, the maturity of debt depends on the relative sensitivity of short-term and long-term risk-free interest rates and long-term default risk. If the long-term, risk-free interest rate is relatively more sensitive to changes in fiscal policies than short-term interest rates, or if default risk is more sensitive than risk-free interest rate, then the debt maturity is shorter.

There are several interesting avenues for future research. First, this paper assumes that the government cannot default within the period once new debt has been issued. The optimal debt policy under lack of commitment and positive default risk implies issuance of a large stock of short-term debt. This in turn increases the likelihood of a self-fulfilling debt crisis if the latter is possible. Therefore, allowing for self-fulfilling debt crises could lead to an interesting trade-off between short-term and long-term debt in such an environment. Second, the government is assumed to be able to default on its debt, but partial default is not allowed in this model. Therefore, it would be interesting to investigate the optimal fiscal and debt policies if the government issues nominal debt that can be inflated away, rather than real debt.

[7] Source: Bloomberg

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Appendix

Proofs

Lemma A.1. *For $k \geq t$*

$$\frac{\partial W_t(\mathbf{s}_t)}{\partial s_{t+k}} = -\beta^k \cdot Pr_t^{t+k} \cdot (\theta_{t+k}\omega'_{t+k} - u'_{t+k}).$$

Proof.

First note that

$$\frac{\partial W_t(\mathbf{s}_t)}{\partial s_t} = -(\theta_t \omega'_t - u'_t) \tag{15}$$

because s_t does not affect budget surpluses or default probabilities in future periods. Then let's show that

$$\frac{\partial W_t(\mathbf{s}_t)}{\partial s_{t+k}} = \beta \cdot Pr_t^{t+1} \cdot \frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+k}}$$

for $k \geq t$. Recall that

$$W_t(\mathbf{s}_t) = u(1-\tau+s_t) + \theta_t \omega(\tau - s_t) + \beta \cdot \mathbb{E}\text{max}\left\{W_{t+1}(\mathbf{s}_{t+1}), V_{t+1}^{def}\right\}$$

and decompose the last component of $W_t(\mathbf{s}_t)$:

$$\mathbb{E}\text{max}\left\{W_{t+1}(\mathbf{s}_{t+1}), V_{t+1}^{def}\right\} = (1 - Pr_t^{t+1}) \cdot \mathbb{E}[V_{t+1}^{def} | V_{t+1}^{def} > W_{t+1}(\mathbf{s}_{t+1})] + Pr_t^{t+1} \cdot W_{t+1}(\mathbf{s}_{t+1})$$

where

$$\mathbb{E}[V_{t+1}^{def} | V_{t+1}^{def} > W_{t+1}(\mathbf{s}_{t+1})] = \frac{1}{1 - F(W_{t+1}(\mathbf{s}_{t+1}))} \cdot \int_{W_{t+1}(\mathbf{s}_{t+1})}^{V^{max}} v\, dF(v) =$$

$$= \frac{1}{1 - Pr_t^{t+1}} \cdot \int_{W_{t+1}(\mathbf{s}_{t+1})}^{V^{max}} v\, dF(v)$$

is the conditional expected value of outside option if the latter is greater than $W_{t+1}(\mathbf{s}_{t+1})$. Then

$$\frac{\partial W_t(\mathbf{s}_t)}{\partial s_{t+k}} = \beta \cdot \frac{\partial\left((1 - Pr_t^{t+1}) \cdot \mathbb{E}[V_{t+1}^{def} | V_{t+1}^{def} > W_{t+1}(\mathbf{s}_{t+1})]\right)}{\partial s_{t+k}} + \beta \cdot \frac{\partial (Pr_t^{t+1} \cdot W_{t+1}(\mathbf{s}_{t+1}))}{\partial s_{t+k}} =$$

$$= \beta \cdot \frac{\partial}{\partial s_{t+1}} \int_{W_{t+1}(\mathbf{s}_{t+1})}^{V^{max}} v\, dF(v) + \beta \cdot \frac{\partial (Pr_t^{t+1} \cdot W_{t+1}(\mathbf{s}_{t+1}))}{\partial s_{t+k}}.$$

Differentiating each term separately yields:

$$\frac{\partial}{\partial s_{t+1}} \int_{W_{t+1}(\mathbf{s}_{t+1})}^{V^{max}} v\, dF(v) = -W_{t+1}(\mathbf{s}_{t+1}) \cdot f(W_{t+1}(\mathbf{s}_{t+1})) \cdot \frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+k}},$$

$$\frac{\partial (Pr_t^{t+1} \cdot W_{t+1}(\mathbf{s}_{t+1}))}{\partial s_{t+k}} = Pr_t^{t+1} \cdot \frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+k}} +$$

$$+W_{t+1}(\mathbf{s}_{t+1}) \cdot f(W_{t+1}(\mathbf{s}_{t+1})) \cdot \frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+k}}$$

because

$$\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}} = \frac{\partial F(W_{t+1}(\mathbf{s}_{t+1}))}{\partial s_{t+1}} = f(W_{t+1}(\mathbf{s}_{t+1})) \cdot \frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+k}}.$$

Thus, combining them we get

$$\frac{\partial W_t(\mathbf{s}_t)}{\partial s_{t+k}} = \beta \cdot Pr_t^{t+1} \cdot \frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+k}}. \tag{16}$$

Iterating the equation (16) forward and using (15) completes the proof:

$$\frac{\partial W_t(\mathbf{s}_t)}{\partial s_{t+k}} = \beta^k \cdot \Pi_{j=1}^{k} Pr_{t+j-1}^{t+j} \cdot \frac{\partial W_{t+k}(\mathbf{s}_{t+k})}{\partial s_{t+k}} =$$

$$= -\beta^k \cdot Pr_t^{t+k} \cdot (\theta_{t+k}\omega'_{t+k} - u'_{t+k}).$$

Proof of Lemma 1.

Following Lemma A.1, the left-hand side of (10) can be rewritten as:

$$\frac{\frac{\partial}{\partial s_{t+1}} W_0(\mathbf{s}_0)}{\frac{\partial}{\partial s_t} W_0(\mathbf{s}_0)} = \frac{\beta^{t+1} \cdot Pr_0^{t+1}}{\beta^t \cdot Pr_0^t} \cdot \frac{\frac{\partial}{\partial s_{t+1}} W_{t+1}(\mathbf{s}_{t+1})}{\frac{\partial}{\partial s_t} W_t(\mathbf{s}_t)} = \frac{\beta \prod_{i=1}^{t+1} F\,(W_i(\mathbf{s}_i))}{\prod_{i=1}^{t} F\,(W_i(\mathbf{s}_i))} \cdot \frac{\frac{\partial}{\partial s_{t+1}} W_{t+1}(\mathbf{s}_{t+1})}{\frac{\partial}{\partial s_t} W_t(\mathbf{s}_t)}$$

$$= \beta Pr_t^{t+1} \cdot \frac{\frac{\partial}{\partial s_{t+1}} W_{t+1}(\mathbf{s}_{t+1})}{\frac{\partial}{\partial s_t} W_t(\mathbf{s}_t)} = \beta Pr_t^{t+1} \frac{\theta_{t+1}\omega'_{t+1} - u'_{t+1}}{\theta_t \omega'_t - u'_t} \tag{17}$$

The denominator of the right-hand side is

$$\frac{\partial}{\partial s_t}(S_0 - B_{-1}^0) = \beta^t \cdot Pr_0^t \cdot \frac{\partial (u'_t \cdot (s_t - b_{-1}^t))}{\partial s_t} +$$

$$+ \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} \beta^j \cdot Pr_0^j \cdot \frac{\frac{\partial Pr_i^{i+1}}{\partial s_t}}{Pr_i^{i+1}} \cdot u'_j \cdot (s_j - b_{-1}^j) \tag{18}$$

The second term of (18) shows how the value of contingent budget surpluses and outstanding debt changes due to a change in default risk.

Note that for $i < t$

$$\frac{\partial Pr_i^{i+1}}{\partial s_t} = f(W_{i+1}(\mathbf{s}_{i+1})) \cdot \frac{\partial W_{i+1}(\mathbf{s}_{i+1})}{\partial s_t} = \beta^{t-1-i} \cdot Pr_{i+1}^t \cdot f(W_{i+1}(\mathbf{s}_{i+1})) \cdot \frac{\partial W_t(\mathbf{s}_t)}{\partial s_t} \tag{19}$$

and for $i \geq t$ the above derivative is just 0.

Plugging in (19) to (18) leads to

$$\frac{\partial}{\partial s_t}(S_0 - B_{-1}^0) = \beta^t \cdot Pr_0^t \cdot \frac{\partial (u'_t \cdot (s_t - b_{-1}^t))}{\partial s_t} +$$

$$+ \frac{\partial W_t(\mathbf{s}_t)}{\partial s_t} \sum_{j=1}^{t} \sum_{i=0}^{j-1} \frac{\beta^{t-1-i} \cdot Pr_{i+1}^t \cdot f(W_{i+1}(\mathbf{s}_{i+1}))}{Pr_i^{i+1}} \cdot \beta^j \cdot Pr_0^j \cdot u'_j \cdot (s_j - b_{-1}^j) +$$

$$+ \frac{\partial W_t(\mathbf{s}_t)}{\partial s_t} \sum_{j=t+1}^{\infty} \sum_{i=0}^{t-1} \frac{\beta^{t-1-i} \cdot Pr_{i+1}^t \cdot f(W_{i+1}(\mathbf{s}_{i+1}))}{Pr_i^{i+1}} \cdot \beta^j \cdot Pr_0^j \cdot u'_j \cdot (s_j - b_{-1}^j) \tag{20}$$

Updating (20) one period ahead yields

$$\frac{\partial}{\partial s_{t+1}}(S_0 - B_{-1}^0) = \beta^{t+1} \cdot Pr_0^{t+1} \cdot \frac{\partial (u'_{t+1} \cdot (s_{t+1} - b_{-1}^{t+1}))}{\partial s_{t+1}} +$$

$$+\frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+1}}\sum_{j=1}^{t+1}\sum_{i=0}^{j-1}\frac{\beta^{t-i}\cdot Pr_{i+1}^{t+1}\cdot f(W_{i+1}(\mathbf{s}_{i+1}))}{Pr_i^{i+1}}\cdot\beta^j\cdot Pr_0^j\cdot u_j'\cdot(s_j-b_{-1}^j)+$$

$$+\frac{\partial W_{t+1}(\mathbf{s}_t)}{\partial s_{t+1}}\sum_{j=t+2}^{\infty}\sum_{i=0}^{t}\frac{\beta^{t-i}\cdot Pr_{i+1}^{t+1}\cdot f(W_{i+1}(\mathbf{s}_{i+1}))}{Pr_i^{i+1}}\cdot\beta^j\cdot Pr_0^j\cdot u_j'\cdot(s_j-b_{-1}^j).$$

The sums $\sum_{j=1}^{t+1}\sum_{i=0}^{j-1}$ and $\sum_{j=t+2}^{\infty}\sum_{i=0}^{t}$ can be rewritten as $\sum_{j=1}^{t}\sum_{i=0}^{j-1}$ and $\sum_{j=t+1}^{\infty}\sum_{i=0}^{t}$, and then we can split $\sum_{i=0}^{t}$ to $\sum_{i=0}^{t-1}$ and $i=t$ so that the above derivative can be reformulated as follows:

$$\frac{\partial}{\partial s_{t+1}}\left(S_0-B_{-1,0}\right)=\beta^{t+1}\cdot Pr_0^{t+1}\cdot\frac{\partial(u_{t+1}'\cdot(s_{t+1}-b_{-1}^{t+1}))}{\partial s_{t+1}}+$$

$$+\beta\cdot Pr_t^{t+1}\frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+1}}\sum_{j=1}^{t}\sum_{i=0}^{j-1}\frac{\beta^{t-1-i}\cdot Pr_{i+1}^{t+1}\cdot f(W_{i+1}(\mathbf{s}_{i+1}))}{Pr_i^{i+1}}\cdot\beta^j\cdot Pr_0^j\cdot u_j'\cdot(s_j-b_{-1}^j)+$$

$$+\beta\cdot Pr_t^{t+1}\frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+1}}\sum_{j=t+1}^{\infty}\sum_{i=0}^{t-1}\frac{\beta^{t-1-i}\cdot Pr_{i+1}^{t+1}\cdot f(W_{i+1}(\mathbf{s}_{i+1}))}{Pr_i^{i+1}}\cdot\beta^j\cdot Pr_0^j\cdot u_j'\cdot(s_j-b_{-1}^j)+$$

$$+\frac{f(W_{t+1}(\mathbf{s}_{t+1}))}{Pr_t^{t+1}}\cdot\frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+1}}\sum_{j=t+1}^{\infty}\beta^j\cdot Pr_0^j\cdot u_j'\cdot(s_j-b_{-1}^j)\tag{21}$$

Note that the ratio of the sum of the second term and the third term of (21) to the corresponding sum of (20) is $\beta\cdot Pr_t^{t+1}\cdot\frac{\frac{\partial}{\partial s_{t+1}}W_{t+1}(\mathbf{s}_{t+1})}{\frac{\partial}{\partial s_t}W_t(\mathbf{s}_t)}$ which is exactly the left-hand side of (10) according to (17).

To see how this cumbersome expression can be simplified, note that if

$$k=\frac{a+b}{c+d}$$

where a, b, c, d and k are just some numbers so that $k=\frac{a}{c}$, then it is true that

$$k=\frac{b}{d}.$$

Therefore, the optimality condition (10) can be simplified as follows:

$$\beta\cdot Pr_t^{t+1}\cdot\frac{\theta_{t+1}\omega_{t+1}'-u_{t+1}'}{\theta_t\omega_t'-u_t'}=$$

$$= \frac{\beta^{t+1}\cdot Pr_0^{t+1}\cdot\frac{\partial(u'_{t+1}\cdot(s_{t+1}-b_{-1}^{t+1}))}{\partial s_{t+1}} + \frac{f(W_{t+1}(\mathbf{s}_{t+1}))}{Pr_t^{t+1}}\cdot\frac{\partial W_{t+1}(\mathbf{s}_{t+1})}{\partial s_{t+1}}\sum_{j=t+1}^{\infty}\beta^j\cdot Pr_0^j\cdot u'_j\cdot(s_j-b_{-1}^j)}{\beta^t Pr_0^t\cdot\frac{\partial(u'_t\cdot(s_t-b_{-1}^t))}{\partial s_t}}$$

$$=$$

$$= \beta\cdot Pr_t^{t+1}\frac{\frac{\partial(u'_{t+1}\cdot(s_{t+1}-b_{-1}^{t+1}))}{\partial s_{t+1}} + \frac{\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}}}{Pr_t^{t+1}}\sum_{j=t+1}^{\infty}\beta^{j-t-1}\cdot Pr_{t+1}^j\cdot u'_j\cdot(s_j-b_{-1}^j)}{\frac{\partial(u'_t\cdot(s_t-b_{-1}^t))}{\partial s_t}} =$$

$$= \beta\cdot Pr_t^{t+1}\frac{\frac{\partial(u'_{t+1}\cdot(s_{t+1}-b_{-1}^{t+1}))}{\partial s_{t+1}} + \frac{\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}}}{Pr_t^{t+1}}(S_{t+1}-B_{-1}^{t+1})}{\frac{\partial(u'_t\cdot(s_t-b_{-1}^t))}{\partial s_t}}.$$

Canceling $\beta\cdot Pr_t^{t+1}$ from both sides yields (11).

Proof of Lemma 2.

Rewrite (12) as follows:

$$\frac{\theta_{t+1}\omega'_{t+1}-u'_{t+1}}{\theta_t\omega'_t-u'_t} =$$

$$\frac{\frac{\partial(u'_{t+1}\cdot(s_{t+1}-b_{-1}^{t+1}))}{\partial s_{t+1}} + \frac{\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}}}{Pr_t^{t+1}}(S_{t+1}-B_{-1}^{t+1}) + \frac{\partial(u'_{t+1}\cdot(b_{-1}^{t+1}-b_{T-1}^{t+1}))}{\partial s_{t+1}} + \frac{\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}}}{Pr_t^{t+1}}(B_{-1}^{t+1}-B_{T-1}^{t+1})}{\frac{\partial(u'_t\cdot(s_t-b_{-1}^t))}{\partial s_t} + \frac{\partial(u'_t\cdot(b_{-1}^t-b_{T-1}^t))}{\partial s_t}}.$$

Recall that

$$\frac{\partial(u'_{t+1}\cdot(s_{t+1}-b_{-1}^{t+1}))}{\partial s_{t+1}} + \frac{\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}}}{Pr_t^{t+1}}(S_{t+1}-B_{-1}^{t+1}) = \frac{\theta_{t+1}\omega'_{t+1}-u'_{t+1}}{\theta_t\omega'_t-u'_t}\cdot\frac{\partial(u'_t\cdot(s_t-b_{-1}^t))}{\partial s_t}$$

Then note that (12) holds only if

$$\frac{\partial(u'_{t+1}\cdot(b_{-1}^{t+1}-b_{T-1}^{t+1}))}{\partial s_{t+1}} + \frac{\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}}}{Pr_t^{t+1}}(B_{-1}^{t+1}-B_{T-1}^{t+1}) = \frac{\theta_{t+1}\omega'_{t+1}-u'_{t+1}}{\theta_t\omega'_t-u'_t}\cdot\frac{\partial(u'_t\cdot(b_{-1}^t-b_{T-1}^t))}{\partial s_t}.$$

Lemma A.2. *Suppose* $u = \frac{c^{1-\gamma}}{1-\gamma}$, $\gamma_C > 0$. *If* $u'(1-\tau+s)\cdot s$ *is increasing in* s *for* $s \in (-(1-\tau), \tau)$ *then* $\frac{\partial u'(1-\tau+s)\cdot s}{\partial s}$ *is strictly decreasing in* s.

Proof.

$$\frac{\partial u'(1-\tau+s)\cdot s}{\partial s} = (1-\tau+s)^{-\gamma_C-1}\cdot(1-\tau+s-\gamma_C\cdot s)$$

$\frac{\partial u'(1-\tau+s)\cdot s}{\partial s} \geq 0$ implies $1-\tau+s-\gamma_C\cdot s \geq 0$, therefore,

$$\frac{\partial^2 u'(1-\tau+s)\cdot s}{\partial s^2} = 2\cdot u''(1-\tau+s) + (-1-\gamma_C)u''(1-\tau+s)\cdot\frac{s}{1-\tau+s} =$$

$$= \frac{1}{1-\tau+s}\cdot u''(1-\tau+s)\cdot(1-\tau+(1-\tau+s-\gamma_C\cdot s)) < 0.$$

Lemma A.3. *Suppose there is no initial debt and Assumption 2 is satisfied. Then*

(i) $S_t > 0\ \forall t \geq 1$;

(ii) $s_t \geq s_{t+1}$ *with strict equality only if* $Pr_t^{t+1} = 1$;

(iii) $W_t \leq W_{t+1}$, $Pr_{t-1}^{t} \leq Pr_t^{t+1}\ \forall t \geq 1$.

Proof.

(i) The proof is by contradiction. Let choose a period $t+1$ such that $S_{t+1} \leq 0$ and $s_{t+1} \leq 0$. Note that we can always find such a period because if, for example, $S_j \leq 0$ but $s_j > 0$ then there exists at least one negative budget surplus in the vector $(s_j, s_{j+1}, \dots)$ and given that

$$S_j = u_j'\cdot s_j + \beta\cdot Pr_j^{j+1}\cdot S_{j+1}$$

weakly negative S_j and positive s_j imply that S_{j+1} is negative as well.

Rewrite the optimality condition (11) as follows:

$$\frac{\frac{\partial(u_t'\cdot s_t)}{\partial s_t}}{\omega_t' - u_t'} = \frac{\frac{\partial(u_{t+1}'\cdot s_{t+1})}{\partial s_{t+1}}}{\omega_{t+1}' - u_{t+1}'} + \frac{\frac{\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}}}{Pr_t^{t+1}}\cdot S_{t+1}}{\omega_{t+1}' - u_{t+1}'}.$$

Given that $\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}} \leq 0$ and $S_{t+1} \leq 0$

$$\frac{\frac{\partial (u'_t \cdot s_t)}{\partial s_t}}{\omega'_t - u'_t} \geq \frac{\frac{\partial (u'_{t+1} \cdot s_{t+1})}{\partial s_{t+1}}}{\omega'_{t+1} - u'_{t+1}}$$

yielding that $s_t \leq s_{t+1} \leq 0$ because $\frac{\partial (u' \cdot s)}{\partial s}$ is decreasing in s by Lemma A.2 and $\omega' - u'$ is increasing in s. Negative s_t in turn implies that $S_t \leq 0$.

Proceeding with backward iterations we find that $S_1 \leq 0$ and $s_1 \leq 0$. The optimality condition for period 0 together with weakly negative S_1 implies:

$$\frac{\frac{\partial (u'_0 \cdot s_0)}{\partial s_0}}{\theta_0 \omega'_0 - u'_0} \geq \frac{\frac{\partial (u'_1 \cdot s_1)}{\partial s_1}}{\omega'_1 - u'_1},$$

$$\Rightarrow \frac{\frac{\partial (u'_0 \cdot s_0)}{\partial s_0}}{\omega'_0 - u'_0} > \frac{\frac{\partial (u'_1 \cdot s_1)}{\partial s_1}}{\omega'_1 - u'_1},$$

$$\Rightarrow s_0 < s_1 \leq 0.$$

However, the latter contradicts the budget constraint $u'_0 \cdot s_0 = -\beta \cdot u'_1 \cdot Pr_0^1 \cdot S_1 \geq 0 \Rightarrow s_0 \geq 0$.

(ii) Given that S_t is (weakly) positive, we infer from the optimality condition that

$$\frac{\frac{\partial (u'_t \cdot s_t)}{\partial s_t}}{\omega'_t - u'_t} = \frac{\frac{\partial (u'_{t+1} \cdot s_{t+1})}{\partial s_{t+1}}}{\omega'_{t+1} - u'_{t+1}} + \underbrace{\frac{\frac{\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}}}{Pr_t^{t+1}} \cdot S_{t+1}}{\omega'_{t+1} - u'_{t+1}}}_{\leq 0} \Rightarrow$$

$$\frac{\frac{\partial (u'_t \cdot s_t)}{\partial s_t}}{\omega'_t - u'_t} \leq \frac{\frac{\partial (u'_{t+1} \cdot s_{t+1})}{\partial s_{t+1}}}{\omega'_{t+1} - u'_{t+1}}$$

for any $t \geq 1$. Recall that each side is strictly decreasing in s meaning that $s_t \geq s_{t+1}$. Given that $S_t > 0 \ \forall t \geq 1$, the strict equality is possible only if $\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}} = 0 \Rightarrow Pr_t^{t+1} = 1$.

(iii) Then we can easily show that $W_t \leq W_{t+1}$ and, hence, $Pr_{t-1}^t \leq Pr_t^{t+1}$. Both W_t and W_{t+1} depend only on an infinite vector of budget surpluses and equal each other if the vectors are the same. The vector starting with s_t has all elements weakly larger than the vector starting with s_{t+1}. Then the result follows from Lemma A.1, which shows that

$$\frac{\partial W_t(\mathbf{s}_t)}{\partial s_{t+k}} = -\beta^k \cdot Pr_t^{t+k} \cdot (\theta_{t+k}\omega'_{t+k} - u'_{t+k}) < 0.$$

Finally, $Pr_{t-1}^t = F(W_t)$ is increasing in W_t.

Proof of Proposition 1

(i) For any $\theta_0 \in [1, \bar{\theta}]$ we can find $s \in [0, \bar{s}]$ such that the optimality condition (11) for $t = 0$ is satisfied for $s_1 = s$ and s_0 which is a solution to the budget constraint $u(1 - \tau + s_0) \cdot s_0 = -\beta \cdot \frac{u'(1-\tau+s)\cdot s}{1-\beta}$. All other optimality conditions are satisfied as well.

(ii) Observe that there is no $s_1 \in [0, \bar{s}]$ that would satisfy the optimality condition (11) for $t = 0$ given that $s_t = s_1\ \forall t \geq 2$ and s_0 is a solution to the budget constraint $u(1 - \tau + s_0) \cdot s_0 = -\beta \cdot \frac{u'(1-\tau+s_1)\cdot s_1}{1-\beta}$. Thus, $s_1 > \bar{s}$.

Then notice that $s_t > \bar{s}$ for all $t \geq 1$. Let by contradiction, $s_t \leq \bar{s}$ for some period t. By Lemma A.3 all future budget surpluses are weakly lower implying that there is no default risk and $\frac{\partial Pr_{t-1}^t}{\partial s_t} = 0$. Then the optimality condition for $t - 1$ implies that $s_{t-1} = s_t$. Iterating backward we would conclude that $s_1 \leq \bar{s}$, which is a contradiction.

Proof of Proposition 2

According to Proposition 1, if $\theta_0 > \bar{\theta}$ then $s_1 > s_2 > \ldots > \bar{s}$ and $Pr_0^1 < Pr_1^2 < \ldots < 1$. The condition (13) can be rewritten as follows

$$\frac{\omega'_{t+1} - u'_{t+1}}{\omega'_t - u'_t} = \frac{u''_{t+1} \cdot b_T^{t+1} + \frac{\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}}}{Pr_t^{t+1}} \cdot B_T^{t+1}}{u''_t \cdot b_T^t}. \tag{22}$$

First, let us show that $B_T^{t+1} \geq 0\ \forall t \geq T \geq 0$. Suppose by contradiction that there is some t such that $B_T^{t+1} < 0$ and $b_T^{t+1} < 0$. Then from (22) we infer that $b_T^t < 0$ and $B_T^t = b_T^t + \beta \cdot Pr_t^{t+1} \cdot B_T^{t+1} < 0$. Iterating backwards yields $B_T^{T+1} < 0$ but it contradicts the budget constraint $B_T^{T+1} = S_{T+1} > 0$.

Then observe that $b_T^t > 0\ \forall t > T \geq 0$. By contradiction, let $b_T^t \leq 0$. Then from (22) we infer $b_T^{t+1} \leq 0$. Thus, we would find $b_T^j \leq 0$ for all $j \geq t$ but this contradicts $B_T^t \geq 0$.

Finally, notice that the left-hand side of (22) is less than one because $s_t > s_{t+1}$ and $\omega' - u'$ is increasing in s. The right-hand side is less than one only if $b_T^t > b_T^{t+1}$ given that $|u''_{t+1}| > |u''_t|$ and $\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}} < 0$.